Motion in uniform $E$ and $H$ fields

Hall effect and magnetoresistance

$$\dot{\vec{k}} = -e \left[ \vec{E} + \frac{\vec{E} \times \vec{H}}{c} \right]$$

$$\Rightarrow \quad \vec{A} \times \dot{\vec{k}} = -e \vec{A} \times \vec{E} - e \frac{\vec{E}}{c} \vec{H}$$

$$\dot{\vec{r}} = -\frac{e c}{e H} \vec{A} \times \dot{\vec{k}} + \vec{w} \quad \vec{w} = \frac{e F}{H} (\vec{E} \times \vec{H})$$

Motion is as before, but with drift velocity $\vec{w}$ added.

To determine orbits in $k$ space, note:

$$\dot{\vec{k}} = -e \vec{E} - \frac{e}{c} \left( \frac{\vec{E}}{H} \right) \times \vec{H}$$

Write $\vec{E} = -(\vec{E} \times \vec{H}) \times \vec{H}$

true when $\vec{E} \perp \vec{H}$

$$\Rightarrow \quad \dot{\vec{z}} = \frac{e}{c} \left( \frac{\vec{E}}{H} \right) \times \vec{H} \quad \vec{z} = e - \vec{t} \vec{k} \cdot \vec{w}$$

Same as if $\vec{E}$ was absent and band structure replaced by

$$\bar{E} (k) = E(k) - \vec{k} \cdot \vec{w}$$

Orbits are intersections of surface of constant $\bar{E}$ with planes $\perp \vec{H}$

We will assume that $-\vec{t} \vec{k} \cdot \vec{w}$ small enough so that $\bar{E} (k)$ is closed (open) so is the constant $\bar{E} (k)$ surface. Good approx in most cases - see text for estimate of numbers.
in nearly free electron model

\[ \varepsilon(k) = \frac{\hbar^2 k^2}{2m} \]

surface of constant energy \( \varepsilon \)

is a sphere of radius

\[ \sqrt{\frac{2m\varepsilon}{\hbar^2}} = k \quad \text{in k-space} \]

\[ \varepsilon(k) = \frac{\hbar^2 k^2}{2m} - \mathbf{\overline{w}} \cdot \mathbf{k} \quad \text{surface of constant } \varepsilon \]

is given by

\[ \frac{\hbar^2}{2m} \left( k - \frac{\overline{w}}{\hbar} \right)^2 = \varepsilon + \frac{1}{2} m \overline{w}^2 \]

sphere in k-space of radius

\[ k = \sqrt{\frac{2m}{\hbar^2} \left( \varepsilon + \frac{1}{2} m \overline{w}^2 \right)} \]

centered about \( \mathbf{k}_0 = \frac{\overline{w}}{\hbar} \)

surface of constant \( \varepsilon \)

shifted by \( \frac{\overline{w}}{\hbar} \) term in direction \( \frac{\overline{w}}{\hbar} \)
Hall effect: \[ \frac{\dot{J}}{\dot{z}} = -\frac{e}{\hbar} \hat{A} \times \hat{k} + \dot{\omega}, \quad \dot{\omega} = \frac{\mathbf{e} \mathbf{E}}{\mathbf{H}} (\mathbf{e} \times \mathbf{H}) \]

current in plane \( \perp \) to \( \mathbf{H} \) is

\[
\bar{J} = ne \langle \mathbf{e} \mathbf{E} \rangle - ne \langle \mathbf{r} \cdot \mathbf{H} \rangle
\]

\[
\dot{J} = -ne \dot{\omega} + ne \frac{\hbar e}{\mathbf{e} \times \mathbf{H}} \mathbf{e} \mathbf{H} \langle \dot{\mathbf{k}} \rangle
\]

where \( \langle \mathbf{r} \cdot \mathbf{H} \rangle \) is steady state average over all occupied electron orbits and over collisions.

**Case (1)** All occupied (or unoccupied) orbits are closed.
we have, after averaging over the electron emerging from the collision at $t=0$, \( \langle \vec{E}(0) \rangle = 0 \). So \( \langle \vec{E} \rangle = \vec{E}(t_0)/t_0 \).

We now average over the time until the second collision, \( \langle t_0 \rangle = T \) (this time is distributed randomly with average equal to \( T \)). Since \( w_c T \gg 1 \), the electron makes many orbits between collisions, \( \vec{E}(t_0) \) when averaged over collision time \( t_0 \) can be equally likely to lie anywhere along the closed orbit.

\[ \Rightarrow \langle \vec{E}(t_0) \rangle = (\text{average } \vec{E} \text{ on orbit}). \]

If electric field \( \vec{E} = 0 \), then (average \( \vec{E} \) on orbit) = 0 also. But when \( \vec{E} \neq 0 \), (average \( \vec{E} \) on orbit) \( \sim m^* \vec{w}/\hbar \). To see this, use effective mass approximation, \( \bar{E}(k) = \frac{\hbar^2 k^2}{2m^*} \).

Then orbit lies on curve of constant \( \bar{E}(k) = \bar{E}(k) - \frac{\hbar^2 k^2}{2m^*} \), which lies on sphere centered at \( \bar{k}_0 = \frac{m^* \vec{w}}{\hbar} \). So (average \( \vec{E} \) on orbit) = \( \langle \vec{E}(t_0) \rangle = \bar{k}_0 \)

\[ \Rightarrow \langle \vec{E} \rangle = \langle \bar{k}(t_0) \rangle = \bar{k}_0 = \frac{m^* \vec{w}}{\hbar T} \cdot \]

So contribution of \( \langle \bar{k} \rangle \) term to current is

\[ \frac{ne \hbar c}{eH} \frac{\hat{A} \times m^* \vec{w}}{\hat{n} \tau} = \frac{ne \hbar \times \vec{w}}{w_c \tau} \]

smaller than drift contribution to current

\[ \vec{j} \approx -ne \vec{w} \text{ by a factor } \frac{1}{w_c T} < 1 \]

So \( \vec{j} \approx -ne \vec{w} \) gives just the drift velocity \( \vec{w} \) in high field limit.
In this case \( \mathbf{J} || \mathbf{H} \Rightarrow \mathbf{J} \perp \mathbf{E} \) and \( \mathbf{H} \Rightarrow \) Lorentz force so strong that electrons move \( \perp \mathbf{E} \) and do not acquire any energy from the \( \mathbf{E} \)-field.

The Hall coefficient in this limit is just \( \frac{E}{jH} \) but in the large \( \mathbf{E} \)-field perpendicular to \( \mathbf{H} \) limit, this is just total \( \mathbf{E} \).

\[
R_{H \rightarrow \infty} = \frac{E}{jH} = \frac{E}{-nece \mathbf{E} \cdot \mathbf{H}} = \frac{1}{nece}
\]

\( \text{Drude value} \)

The above was for closed occupied orbits

If we had closed unoccupied orbits we would use the hole picture to get

\[
R_{H \rightarrow \infty} = +\frac{1}{nece} > 0
\]

(\( n_h \) is density of holes, each hole has charge \( +e \))

If there is more than one partially full band with only closed occupied or unoccupied orbits, then

\[
\mathbf{J} = -\text{neff} \frac{ec}{H} (\mathbf{E} \times \mathbf{H}) \quad \text{where neff} = n - n_h
\]

\[
R_{H \rightarrow \infty} = -\frac{1}{\text{neff} ec}
\]

The effects of holes explains why \( R_{\infty} \) can have non Drude values, and even be \( > 0 \).
See text for what happens when $\text{Meff} = 0$. This is case for undoped semiconductor.

Another way to view this is in terms of conductivity tensor. Keeping contribution to $\mathbf{J}$ from the $\langle \mathbf{f} \rangle$ term gives

$$\mathbf{J} = -ne\mathbf{w} + ne\frac{\mathbf{E}}{\text{H}} \times \mathbf{w} \quad \Rightarrow \quad \mathbf{w} = \frac{e\mathbf{E}}{\mathbf{H}} \times (\mathbf{E} \times \mathbf{H})$$

for $\mathbf{H} = \hat{z}$ direction we have

$$\mathbf{J} = \frac{ne\mathbf{c}}{\mathbf{H}} \left( \mathbf{\hat{z}} \times \mathbf{E} + \frac{1}{\text{H}} \mathbf{E} \times \mathbf{H} \right) = \mathbf{\sigma} \cdot \mathbf{E}$$

with $\mathbf{\sigma} = \frac{ne\mathbf{c}}{\mathbf{H}} \left( \begin{array}{cc} \frac{1}{\text{H}} & -1 \\ 1 & \frac{1}{\text{H}} \end{array} \right)$

or writing $\sigma_0 = \frac{ne^2\mathbf{c}}{\text{m}^*} \frac{\mathbf{m}^* \mathbf{c}}{\mathbf{E} \times \mathbf{H}} = \frac{ne\mathbf{c}}{\mathbf{H}}$ where

$\sigma_0$ is Drude conductivity $\Rightarrow \mathbf{\sigma} = \sigma_0 \left( \begin{array}{cc} \frac{1}{\text{H}} \left( \frac{1}{\text{H}} \right)^2 & - \frac{1}{\text{H}} \\ \frac{1}{\text{H}} & \frac{1}{\text{H}} \left( \frac{1}{\text{H}} \right)^2 \end{array} \right)$

(Compare with prob #1 on HW #1!!)

$\Rightarrow$ resistivity tensor $\mathbf{\rho} = \mathbf{\sigma}^{-1} = \frac{1}{\sigma_0} \left( \begin{array}{cc} \frac{1}{\text{H}} \left( \frac{1}{\text{H}} \right)^2 + \frac{1}{\text{H}} & - \frac{1}{\text{H}} \\ \frac{1}{\text{H}} & \frac{1}{\text{H}} \left( \frac{1}{\text{H}} \right)^2 \end{array} \right)$

\[
\frac{1}{\sigma} = \frac{1}{\sigma_0} \left( \frac{1}{\text{H}} + \frac{1}{\text{H}} \right)^2 \left( \begin{array}{cc} 1 & \frac{1}{\text{H}} \\ \frac{1}{\text{H}} & 1 \end{array} \right) \approx \frac{1}{\sigma_0} \left( \begin{array}{cc} 1 & \frac{1}{\text{H}} \\ -\frac{1}{\text{H}} & 1 \end{array} \right) = \left( \begin{array}{cc} \frac{\mathbf{p} \times \mathbf{f}_0}{\mathbf{p} \times \mathbf{f}_0} \end{array} \right)
\]
\[ \vec{j} = \sigma \cdot \vec{E} \]
\[ \sigma = \frac{e^2}{m^* e^2} \left( \frac{1}{\omega_c} \left( \begin{array}{cc} 1 & -1 \\ 1 & \frac{1}{\omega_c} \end{array} \right) \right) \]
\[ \omega_c = \frac{e}{m^* c} \]

Then \[ \vec{E} = \vec{p} \cdot \vec{j} \]
where \[ \vec{p} = \frac{1}{\sigma} \left( \begin{array}{cc} 1 & - \omega_c \end{array} \right) = \left( \begin{array}{c} p_x \ \ p_y \end{array} \right) \]

For \[ \vec{f} = \vec{j} \times \vec{E} \] then \[ E_y = \vec{p}_y \vec{j} = -p_x j_y \]

Hall coef: \[ R = \frac{E_y}{\vec{j} \cdot \vec{H}} = \frac{-p_x j_y}{\vec{p} \cdot \vec{H}} = \frac{-\omega_c}{\sigma_0 H} = \frac{-eH}{m^* c \cdot ne^2 c H} \]

\[ = -\frac{1}{m^* c} \quad \text{Drude value} \]

\[ \text{For holes: } \vec{f} = m_h \]

\[ \text{For electrons we used } \vec{f} = -me \vec{v} + me \frac{\vec{H} \times \vec{v}}{\omega_c} \]

\[ \text{For holes we use instead } \vec{f} = +me \vec{v} - me \frac{\vec{H} \times \vec{v}}{\omega_c} \]

Since charge carriers have charge \( e \).

All results carry through except take \( e \to -e \)

\[ \Rightarrow R = \frac{1}{m^* e c} \]
(ideal coefficients in)

\[ R = -\frac{\rho_{xy}}{H} \] (see quantum Hall effect notes)

\[ = -\frac{\mu e^2}{8\pi^2 \hbar^2} \] 

\[ = -\frac{eH}{m^2} \] 

\[ = -\frac{1}{m^2} \] as before.

Magnetic resistance

\[ \rho_{xx} - \rho_{xy} = \frac{1}{\sigma} \]

saturates to finite value as \( H \to 0 \)

just as was found in Drude model, except now \( \sigma \) is \( m^2 \) eff. there are several partially filled bands.

**Case 2**

Neither all occupied states, nor all unoccupied states, have closed orbits in either electron or hole picture. There are open orbits, we have to consider.

Now we will find that the \( \langle \hat{\mathbf{k}} \rangle \) contribution to current \( \hat{j} \) from these open orbits no longer vanishes in the \( \mathcal{W} \to \infty \) limit, and it dominates over the drift contribution to the current -new.

When \( \hat{E} = 0 \), \( \hat{H} = H \hat{z} \) induces motion in orbits on the constant-energy surfaces. An electron moving in an open orbit in \( k \)-space in the \( +\hat{k}_y \) direction, gives a current in real space in the \( +\hat{x} \) direction (rotate by 90° about \( \hat{z} \)). However when \( \hat{E} = 0 \), each occupied open orbit going in one direction is paired with an occupied open orbit going in the opposite direction, so the net current is zero.
Note: For an open orbit traveling along $\hat{\mathbf{k}}_y$, $k_y(t)$ is periodic in time $\Rightarrow V_y = \langle \frac{\partial E}{\partial k_y} \rangle = 0$ averaged over time. But $k_x(t)$ is constant $\Rightarrow V_x = \langle \frac{\partial E}{\partial k_x} \rangle = 0 \Rightarrow$ electron moves in $\hat{x}$ direction.

In $k$-space

$E_x < 0 \Rightarrow$ net
$\nu_x > 0 \Rightarrow j_x < 0$

$j_x \sim E_x$ $\&$ lowest order in $E$

$\frac{j}{\nu} \sim \hat{x}(\hat{E} \cdot \hat{x})$

When $E \neq 0$, in steady state, there will be an imbalance in occupation of open orbits, so that those orbits which absorb energy from the $E$-field have a larger population than those which lose energy to the field. ($E$-field heats up metal!)

- Open orbits in $+\hat{y}$ direction have real space direction $+\hat{x} \Rightarrow$ they gain energy from $E$-field if $E_x < 0$
- As energy absorbed is $-e^2 \cdot \nu \cdot \tilde{C}$ (between collisions).

Open orbits in $-\hat{y}$ directions have real space direction $-\hat{x} \Rightarrow$ they lose energy if $E_x > 0$.

We assume therefore that the imbalance in occupation of open orbits in steady state gives rise to a current. If $\hat{\mathbf{m}}$ is the direction in real space of the open orbits, then the contribution to the current $\tilde{j}$ is in the $\hat{\mathbf{m}}$ direction, and proportional to some function of $\tilde{E} \cdot \hat{\mathbf{m}}$.

$\tilde{j}_{\text{open}} = m g(E \cdot \hat{m}) \sim \exp(\text{small } \tilde{E})$.
Equivalently, since $\bar{E} = E - \hbar \mathbf{k} \cdot \mathbf{w}$ is conserved between collisions, if $\Delta \mathbf{E} = -e\bar{E} \cdot \mathbf{v} < 0$, energy absorbed by electron from $E$-field then

$$\Delta \mathbf{E} = 0 \Rightarrow \Delta \mathbf{E} = \hbar \mathbf{w} \cdot \mathbf{\Delta k}$$

So again we see in our example

![Diagram of orbits](image)

that it is the **right** hand open orbits moving along $+\mathbf{k}$ that absorb energy, i.e. $\mathbf{w} \cdot \mathbf{\Delta k} > 0$ for these orbits, while $\mathbf{w} \cdot \mathbf{\Delta k} < 0$ for left hand open orbits moving along $-\mathbf{k}$.

![Handwritten notes](image)

So both $\mathbf{w} \cdot \mathbf{\Delta k}$ and $-E \cdot v$ tell how much energy the electron absorbs from $E$-field.
This imbalance in steady state occupation of open orbits is determined by the quantity \(-e \vec{E} \cdot \vec{v}_c\), the energy absorbed by selection from \(\vec{E}\)-field in between collisions.

If \(\vec{A}\) is real space direction of open orbit, \(\Rightarrow <\vec{v}>\hat{\vec{a}}\) in \(\hat{\vec{a}}\) direction, so the current due to open orbits is in the \(\hat{\vec{a}}\) direction, and is some function of \((\vec{E} \cdot \hat{\vec{a}})\)

\[
\vec{j}_{\text{open}} = \hat{\vec{a}} g(\vec{E} \cdot \hat{\vec{a}}) \quad \text{where } g(\vec{E}) = \begin{cases} f = 0 \text{ when } \vec{E} = 0, \text{ and} \\ f(\vec{E}) = -f(-\vec{E}) \end{cases}
\]

\[
\vec{j}_{\text{open}} \sim \hat{\vec{a}} (\vec{E} \cdot \vec{E})
\]

We can write the contribution to conductivity tensor due to open orbits as

\[
\vec{j}_{\text{open}} = \hat{\vec{a}} \vec{E} \quad \text{where } \vec{\sigma} = \chi \sigma_0 \hat{\vec{a}} \hat{\vec{a}}
\]

If we choose \(\hat{\vec{a}}\) in \(\hat{\vec{x}}\) direction,

\[
\vec{\sigma} = \chi \sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

If we treat the contribution to conductivity tensor from closed orbits as before, we get for total conductivity tensor

\[
\vec{\sigma} = \frac{\sigma_0}{(\omega_c T)^2} \begin{pmatrix} 1 - \omega_c T & \omega_c T \\ \omega_c T & 1 \end{pmatrix} + \chi \sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \sigma_0 \begin{pmatrix} \frac{1}{\omega_c T} + \chi (\omega_c T)^2 & -\frac{1}{\omega_c T} \\ \frac{1}{\omega_c T} & \frac{1}{(\omega_c T)^2} \end{pmatrix}
\]
or resistivity tensor \( \vec{\rho} = \sigma^{-1} \)

\[
\sigma^{-1} = \sigma_0 \left[ \frac{1}{(\omega e_2)^2 + \frac{1}{\omega e_2} + \frac{1}{\omega e_2}} \right] \left( \frac{1}{\alpha e_2} \right) \lambda + \frac{1}{\omega e_2^2}
\]

\[
\sigma^{-1} = \frac{1}{\sigma_0 (1+\lambda)} \left( \frac{1}{\omega e_2} \right) \lambda + \frac{1}{\omega e_2^2}
\]

Note \( f_{xy} = f_{-xy} \) as before for closed orbits, and Hall coefficient is \( \frac{-\omega e_2}{\sigma_0 (1+\lambda)} = \frac{-1}{n \omega c (1+\lambda)} \) same as before except for factor \((1+\lambda)\).

But now \( f_{xx} \neq f_{yy} \). We have

\[
\text{exp't } f_{xx} \quad - \quad \text{magneto-resistance for current flowing } \parallel \text{ to open orbits in real space (i.e. } \hat{z} = \hat{e}_x \text{)}
\]

\[
= \frac{\lambda}{\sigma_0 (1+\lambda)} \quad \text{saturates as } H \to \infty \text{ as in Drude mode }
\]

\[
\text{exp't } f_{yy} \quad - \quad \text{magneto-resistance when current flowing } \perp \text{ direction of open orbits in real space (i.e. } \hat{z} = \hat{e}_y \text{)}
\]

\[
= \frac{\lambda}{\sigma_0 (1+\lambda)} (\omega e_2)^2 \sim H^2 \quad \text{does not saturate as } H \to \infty \quad \text{grows as } H^2!
\]

magneto-resistance which keeps increasing with \( H \)
is signal for presence of open orbits on Fermi Surface.
For a current in a general direction \( \mathbf{j} = j_0 (\cos \phi) \mathbf{x} \), where \( \phi \) measures angle from \( \mathbf{x} \), the direction of the open orbits in real space, we have

\[
\frac{\mathbf{E}}{\mathbf{j}} = \frac{j_0}{\sigma_0 (1 + \lambda)} \begin{pmatrix} \cos \phi + (w_c \tau) \sin \phi \\ -(w_c \tau) \cos \phi + (\lambda (w_c \tau)^2 + 1) \sin \phi \end{pmatrix}
\]

and the longitudinal magnetoresistance is

\[
\sigma = \frac{\mathbf{E} \cdot \mathbf{j}}{\mathbf{j}^2}
\]

\[
= \frac{1}{\sigma_0 (1 + \lambda)} \left[ \cos^2 \phi + (w_c \tau) \sin \phi \cos \phi \\ - (w_c \tau) \cos \phi \sin \phi + [\lambda (w_c \tau)^2 + 1] \sin^2 \phi \right]
\]

For a constant, oxide-like part from closed orbits and increases without bound as \( H \) increases from open orbits.