Magnetic properties of Free Electron Gas

In the presence of an applied magnetic field $\vec{H}$, the electron gas will develop a net magnetization via two effects

1) the intrinsic spins of the electrons anti-align with $\vec{H}$ $\Rightarrow$ magnetic moments align with $\vec{H}$ $\Rightarrow$ paramagnetic effect
   Pauli Paramagnetism

2) the electrons move in closed orbits $\Rightarrow$ circulating currents $\Rightarrow$ magnetic moments anti-aligned with $\vec{H}$ $\Rightarrow$ diamagnetic effect
   Landau diamagnetism

We consider first Pauli Paramagnetism
(A+M Chpt 31

An electron with intrinsic spin $\vec{s}$ $(s_z = \pm 1)$ has intrinsic magnetic moment $\vec{\mu} = -\mu_0 \vec{s}$
where $\mu_0 = \frac{e\hbar}{2mc}$ is the Bohr magneton

The interaction energy of the spin with the applied magnetic field is

$$\mathcal{H} = -\vec{\mu} \cdot \vec{H} = \mu_0 \vec{s} \cdot \vec{H}$$

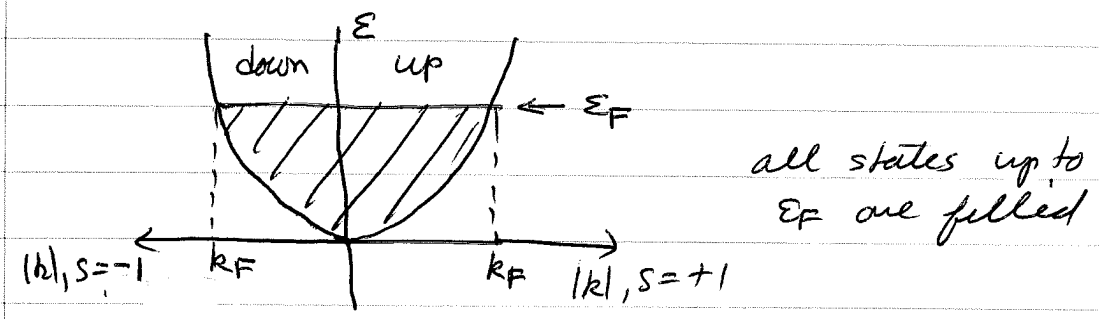
If we take $\vec{H} = H\hat{z}$, the above $\delta\mathcal{H}$ then gives the single electron energy eigenvalues

$$\varepsilon_{ks} = \frac{\hbar^2 k^2}{2m} + s\mu_0 H \qquad \text{with} \quad s = \pm 1$$

(parallel) spin up electrons, $s=+1$, increase their energy
(antiparallel) spin down electrons, $s=-1$, decrease their energy

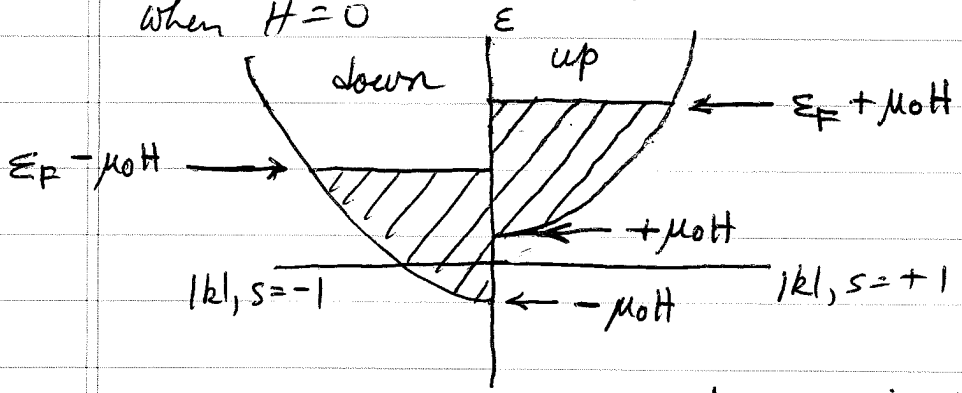
$\Rightarrow$ In equilibrium, this shift of electron energy with s results in a net excess of $s=-1$ electrons

For $H=0$, the dispersion curves look like



all states up to ε_F are filled

Now imagine turning on a small $\vec{H}$, but keeping the $s=\pm 1$ populations of electrons the same as when $H=0$

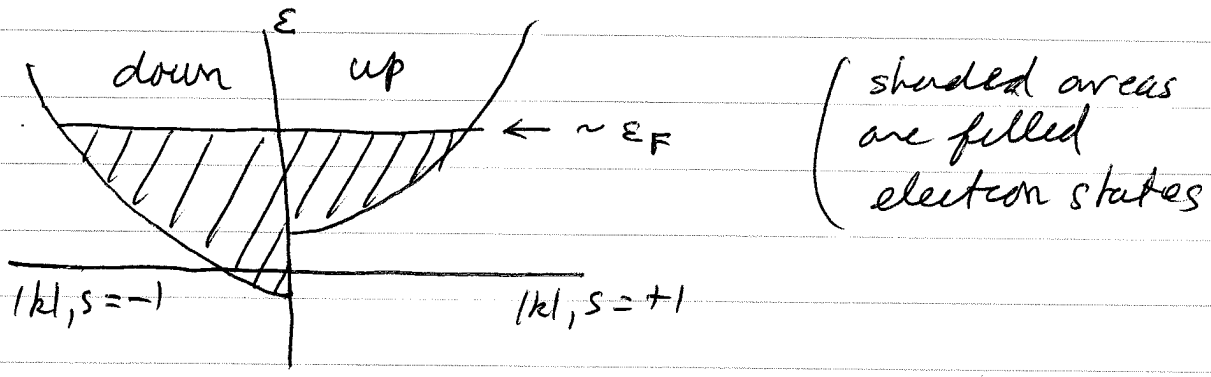


(shaded areas are filled electron states)

We see that above cannot remain the ground

state, as the energy will be lowered by having up electrons at $\varepsilon_F + \mu_0 H$, $s = +1$ convert into down electrons and go into the empty states at $\varepsilon_F - \mu_0 H$, $s = -1$.

The ground state will instead look like

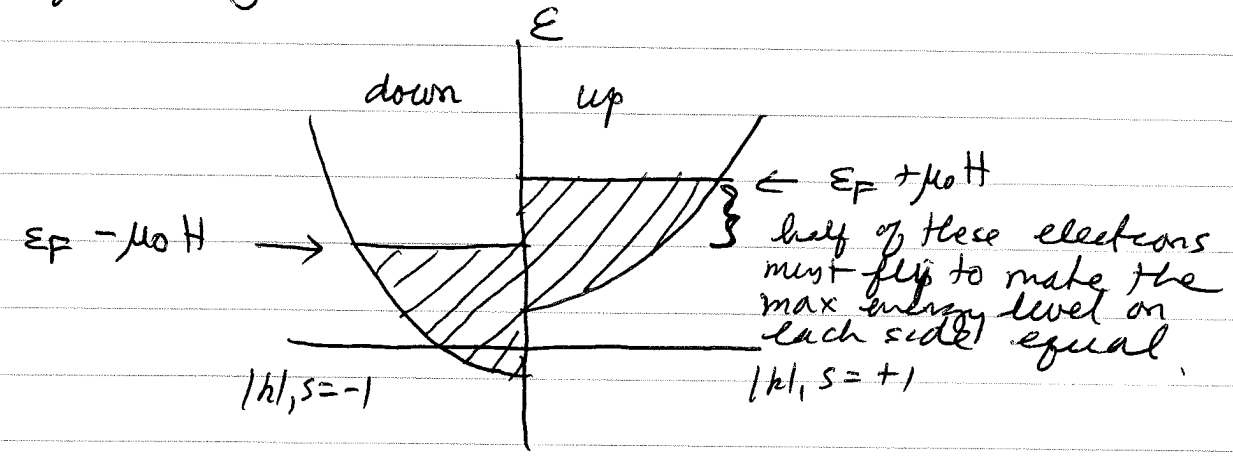


and we thus see that there will be more down than up electrons in the ground state. Since $s = \pm 1$ electrons have magnetic moment $\mu = -\mu_0 s$, the system has a net positive magnetization M aligned with H.

To see how big is M we need to compute the number of up electrons that flip into down electrons when H is turned on.

We will assume that H is small enough that $\mu_0 H \ll \varepsilon_F$. When this is so, we can ignore the fact that the density of states has a slight variation with energy ε over the range $\varepsilon_F + \mu_0 H$ to $\varepsilon_F - \mu_0 H$ and assume it to be roughly constant $g(\varepsilon_F)$.

The number of up electrons that flip is then easily computed from the following sketch



The number of up electrons that must flip is therefore

$$g_+(\varepsilon_F)\,\Delta\varepsilon$$

where $g_+(\varepsilon_F) = \frac{1}{2} g(\varepsilon_F)$ is the density of states of up electrons at ε_F, which is half the total density of states at ε_F, and $\Delta\varepsilon = \frac{1}{2}[\mu_0 H - (-\mu_0 H_0)] = \mu_0 H$ is the energy interval that must flip. The number that flip is therefore

$$\Delta n = \frac{1}{2} g(\varepsilon_F)\,\mu_0 H.$$

In the new ground state, the number of down electrons is now $n_0 + \Delta n$, and the number of up electrons is $n_0 - \Delta n$, where $n_0 = \frac{1}{2} n$ is the ~~num~~ number when $H = 0$.

So the net magnetization is now (at $T=0$)

$$\begin{aligned} M &= \mu_0 (n_- - n_+) \\ &= \mu_0 (n_0 + \Delta n - (n_0 - \Delta n)) \\ &= 2\mu_0 \Delta n \\ &= g(\varepsilon_F) \mu_0^2 H \end{aligned}$$

and the <u>Pauli paramagnetic susceptibility</u> is

$$\boxed{\chi_p = \frac{\partial M}{\partial H} = g(\varepsilon_F) \mu_0^2}$$

proportional to density of states at Fermi energy

For the <u>free electron gas</u> we had

$$g(\varepsilon_F) = \frac{3}{2} \frac{n}{\varepsilon_F}$$

$$\Rightarrow \boxed{\chi_p = \frac{3}{2} \frac{n}{\varepsilon_F} \mu_0^2}$$

$n \sim k_F^3$, $\varepsilon_F \sim k_F^2$, so $\chi_p \sim k_F \sim \frac{1}{(r_s/a_0)}$

$$\chi_p = \frac{2.59}{(r_s/a_0)} \times 10^{-6}$$

corrections to above result at finite T of order $\left(\frac{T}{T_F}\right)^2$ so above is very good ~~but~~ at all $T \ll T_F$ and so good at room temperature.

Compare to experiment

Metal	r_s/a_0	χ_p^{theory}	χ_p^{ext}	$\times 10^{-6}$
Li	3.25	0.80	2.0	
Na	3.93	0.66	1.1	
K	4.86	0.53	0.8	
Rb	5.20	0.50	0.8	
Cs	5.62	0.46	0.8	

turns out that the disagreement between theory and expt is mainly due to having neglected electron-electron interactions!

Note that χ_p above is very different from what one gets with classical statistics.

Classically $M \sim \left[\frac{(+1)e^{-\mu_0 H/k_B T} + (-1)e^{+\mu_0 H/k_B T}}{e^{-\mu_0 H/k_B T} + e^{+\mu_0 H/k_B T}}\right](-\mu_0)n$

$$= \left[\frac{e^{\mu_0 H/k_B T} - e^{-\mu_0 H/k_B T}}{e^{\mu_0 H/k_B T} + e^{\mu_0 H/k_B T}}\right]\mu_0 n$$

$$\sim \frac{\mu_0^2 H n}{k_B T} \quad \text{when } \mu_0 H \ll k_B T$$

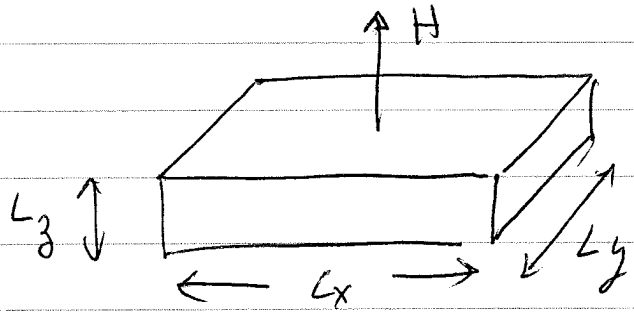
$$\chi_{classical} = \frac{dM}{dH} \sim \frac{\mu_0^2 n}{k_B T} \quad \text{Curie law} \sim \frac{1}{T}$$

So $\frac{\chi_p}{\chi_{classical}} \sim \left(\frac{T}{T_F}\right) \ll 1$

Landau Diamagnetism - Landau Levels

Here we wish to consider the effect of the magnetic field on the orbital motion of the conduction electrons. To do so we must solve the quantum mechanical problem of a charged particle moving in a uniform magnetic field.

The geometry we consider is



uniform $\vec{H} = H\hat{z}$

For a particle of charge q in a static uniform magnetic field, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2$$

where $\vec{A}$ is the vector potential, $\vec{H} = \vec{\nabla} \times \vec{A}$

$q = -e$ is the charge of the electron

For $\vec{H} = H\hat{z}$ we will use $\vec{A} = -yH\hat{x}$

Substitute these into $\mathcal{H}$ to get:

$$\mathcal{H} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}yH\hat{x}\right)^2$$

$$= \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial z^2} - \hbar^2\frac{\partial^2}{\partial y^2} + \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c}Hy\right)^2\right]$$

We want to find the eigenstates ψ that solve

$$\mathcal{H}\psi = \varepsilon\psi \qquad \varepsilon \text{ is eigenvalue of energy}$$

Try solution of the form

$$\psi(x,y,z) = e^{ik_x x}\, e^{ik_z z}\, \phi(y)$$

This form is suggested as $\mathcal{H}$ is translationally invariant in x and z, but not in y (due to our particular choice for $\vec{A}$)

Substitute this ψ into above Schrodinger Equation to get

$$\frac{1}{2m}\left[\hbar^2 k_z^2 - \hbar^2\frac{\partial^2}{\partial y^2} + \left(\hbar k_x - \frac{e}{c}Hy\right)^2\right]\phi(y) = \varepsilon\phi(y)$$

or

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2m}\left(\hbar k_x - \frac{e}{c}Hy\right)^2\phi = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m}\right)\phi$$

define y_0 such that $\hbar k_x \equiv \frac{eH}{c}y_0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2m}\left(\frac{eH}{c}\right)^2(y-y_0)^2\phi = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m}\right)\phi$$

define cyclotron frequency $\omega_c \equiv \frac{eH}{mc}$

(a classical charged particle in uniform $\vec{H}$ moves in a circular orbit with angular velocity ω_c)

Finally we get

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega_c^2 (y-y_0)^2 \right] \phi(y) = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m} \right) \phi(y)$$

This is just the Hamiltonian for a single harmonic oscillator of frequency ω_c that is centered at $y = y_0$.

We know the eigenvalues of energy of the harmonic oscillator are just

$$\hbar\omega_c (n+1/2) \qquad n = 0, 1, 2, \ldots$$

So we then have

$$\varepsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_c (n+1/2)$$

or the energy eigenvalues of our particle are

$$\varepsilon = \underbrace{\frac{\hbar^2 k_z^2}{2m}}_{\text{kinetic energy of motion along } z \text{ parallel to } \vec{H}} + \underbrace{\hbar\omega_c (n+1/2)}_{\text{kinetic energy of orbital motion in } xy \text{ plane} \perp \text{to } \vec{H}}$$

The wavefunctions $\phi_n(y)$ are the usual harmonic oscillator wavefunctions (gaussian × Hermite polynomial) only centered at y_0.

We can therefore write our solution in terms of 3 quantum numbers, k_x, k_z, n

$$\begin{cases} \psi_{k_x,k_z,n}(x,y,z) = e^{ik_x x} e^{ik_z z} \tilde{\phi}_n(y-y_0) \\ \qquad\qquad (\tilde{\phi} \text{ is h.o. wavefunction centered at origin}) \\ E(k_x,k_z,n) = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega_c(n+1/2) \end{cases}$$

where $y_0 = \hbar k_x \frac{c}{eH} = \frac{\hbar k_x}{m\omega_c}$

Note E is independent of k_x so for fixed k_z and n there are many degenerate states corresponding to the different possible choices for k_x.

What are the possible values of k_x?

If we take periodic boundary conditions along x, $\psi(x+L_x, y, z) = \psi(x, y, z)$ then we must have

$$e^{ik_x x} = e^{ik_x(x+L_x)} \Rightarrow k_x = \frac{2\pi}{L_x} \times (\text{integer})$$

But k_x also determines the value of y_0 about which the wavefunction is centered in the y-direction. Therefore we must have

$$0 \le y_0 \le L_y \Rightarrow 0 \le \frac{\hbar k_x}{m\omega_c} \le L_y$$

$$\Rightarrow k_{x\,max} = L_y \frac{m\omega_c}{\hbar} = \frac{L_y}{\hbar} \frac{meH}{mc} = \frac{L_y eH}{\hbar c}$$

Combining these two conditions we have the the number of allowed values k_x can take is given by

$$\frac{k_{x\,max}}{\Delta k_x} = \frac{L_y \frac{eH}{\hbar c}}{\left(\frac{2\pi}{L_x}\right)} = L_x L_y \frac{eH}{2\pi\hbar c} = L_x L_y \frac{eH}{\hbar c}$$

$$= \frac{L_x L_y H}{\left(\frac{hc}{e}\right)}$$

to get the numbe of allowed electron states with energy $\frac{\hbar k_z^2}{2m} + \hbar\omega_c(n+1/2)$ we should multiply above by a factor of 2 for the two posible spin states.

$$\text{Degeneracy } W = \frac{2 L_x L_y H}{\left(\frac{hc}{e}\right)} = \frac{\Phi}{\left(\frac{hc}{2e}\right)} = \frac{\Phi}{\Phi_0}$$

where $\Phi = L_x L_y H$ is the total magnetic flux penetrating the system, and

$\Phi_0 = \frac{hc}{2e}$ has units of magnetic flux and is called the "flux quantum"

$$\Phi_0 = 2.07 \times 10^{-7} \text{ gauss-cm}^2$$

degeneracy is $\frac{\Phi}{\Phi_0}$ = number of flux quanta

Consider now just the motion of the electron in the xy plane. The energy of this motion is

$$\tilde{\varepsilon} = \varepsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_c (n+1/2) \qquad n = 0, 1, 2, \ldots$$

The states corresponding to a given value of n are called the "n^{th} Landau Level". The n^{th} landau level has a degeneracy of Φ/Φ_0, or equivalently, the number of electrons per unit area that one can put into a given Landau level is

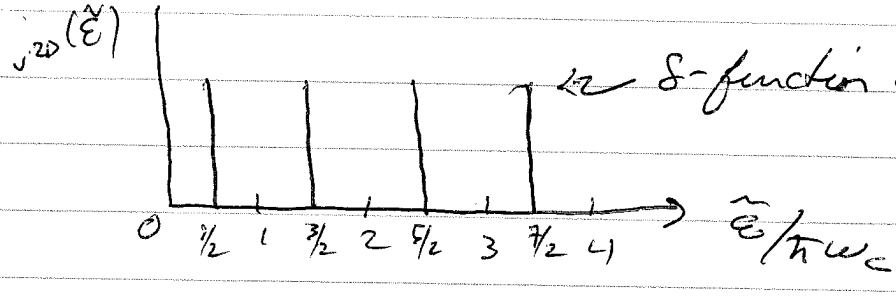
$$\frac{1}{L_x L_y}\frac{\Phi}{\Phi_0} = \frac{H}{\Phi_0}$$

We can summarize this by giving the <u>density of states</u> for the energy $\tilde{\varepsilon}$ in the xy plane

$g_{2D}(\tilde{\varepsilon})\,\Delta\varepsilon$ = number of electron states per unit area with energy in the range $\tilde{\varepsilon}$ to $\tilde{\varepsilon} + \Delta\varepsilon$.

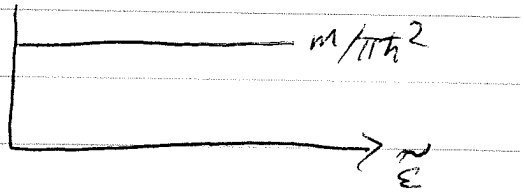
Since there are only states at the discrete energy values $\hbar\omega_c(n+1/2)$, $g_{2D}(\tilde{\varepsilon})$ is a sum of δ-functions at these discrete values - the amplitude of each δ-function is just the degeneracy per area H/Φ_0

$$g_{2D}(\tilde{\varepsilon}) = \sum_n \frac{H}{\Phi_0}\,\delta(\tilde{\varepsilon} - \hbar\omega_c(n+1/2))$$



We can compare this to the 2D density of states when $H=0$. From problem (3b) of HW set 1 you will find that at $H=0$, $g_{2D}(\tilde{\varepsilon})$ is a constant

$H=0$: $g_{2D}(\tilde{\varepsilon}) = \frac{m}{\pi\hbar^2}$



To compare $H=0$ with $H>0$, consider computing the average density of state for $H>0$ where we average over an energy interval large compared to the spacing between the Landau levels $\hbar\omega_c$.

average density of states $\bar{g} = \frac{(\text{\# } \delta\text{-function spikes in } \Delta\varepsilon) \times \frac{H}{\Phi_0}}{\text{interval width } \Delta\varepsilon}$

If we take $\Delta\varepsilon = M\hbar\omega_c$ for a large integer M, then on average there will be M δ-function spikes in this interval, so

$$\bar{g} = \frac{M \times \frac{H}{\Phi_0}}{M\hbar\omega_c} = \frac{H}{\left(\frac{hc}{2e}\right)} \frac{1}{\hbar\left(\frac{eH}{mc}\right)} = \frac{m}{\pi\hbar^2}$$

so average density of state at $H>0$

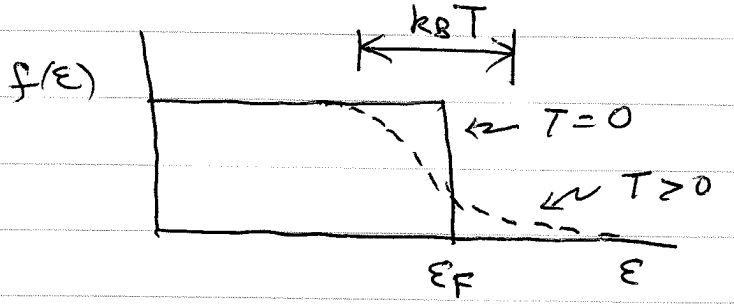
$\bar{g} = \frac{m}{\pi\hbar^2}$ = constant density of states at $H=0$

So turning on the magnetic field bunches the energy eigenstates up into discrete levels, but the average number of states per unit energy remains the same (provided we average on interval $\gg \hbar\omega_c$)

Suppose we had an actual 2D electron gas. One can think of making this in a thin metallic film or a semiconductor inversion layer where the gas is confined to a region in space along $\hat{z}$ so small that only the lowest allowed value of k_z is occupied, ie $\frac{2\pi}{L_z} = \Delta k_z$ gives $\frac{\hbar^2(\Delta k_z)^2}{2m}$ larger than all other energy scales.

What is necessary so that one could detect the difference between the discrete Landau level structure at finite $H > 0$, and the average density of states which is equal to its $H = 0$ value?

If f is the Fermi function, we know that finite temperature smears out the sharp cutoff at $\varepsilon = \varepsilon_F$ that exists at $T = 0$.



To see the Landau level structure we thus need this smearing to be small on the scale of the spacing between the Landau levels

ie. need $k_B T \ll \hbar \omega_c$

using $\omega_c = \frac{eH}{mc}$ and m the free electron mass one can compute

$$\omega_c = 1.76 \times 10^{11} \text{ sec}^{-1}$$ for a $H = 1$ tesla $= 10^4$ gauss magnetic field.

1 tesla is a big field. In a laboratory setup such as in BL one can buy a 10 tesla magnet. Larger field strengths require specialized facilities

So for $H = 1$ tesla, $\boxed{\frac{\hbar\omega_c}{k_B} = 1.34\,^\circ K}$ ☆

So in a 1 tesla field one needs to go well below $1^\circ K$ to see Landau level structure. In a 10 tesla field one needs to go well below $10^\circ K$. So quite low temperatures are needed.

There is a second condition. In solving Schrodinger's equation for the Landau levels, we ignored any sources of electron scattering (scattering off phonons, plasmons, lattice impurities, etc.) If τ is the scattering time, including such scattering generally leads, via the uncertainty principle, to a broadening of the energy levels of the eigenstates to a finite width $\delta E \sim \frac{\hbar}{\tau}$

So to see Landau level structure we need

$$\delta\varepsilon \ll \hbar\omega_c \Rightarrow \frac{\hbar}{\tau} \ll \hbar\omega_c$$

$$\Rightarrow \omega_c\tau \gg 1$$

using $\omega_c = 1.76 \times 10^{11}\ \text{sec}^{-1}$ in $H = 1$ tesla and from resistivity measurements used to estimate τ from Drude's model we get

room temp $\tau \sim 10^{-14}$ sec, $\omega_c\tau \sim 0.00176$

77°K (liquid N_2) $\tau \sim 10^{-13}$ sec, $\omega_c\tau \sim 0.0176$

we again see that we will need very low temperatures (large τ) to get $\omega_c\tau \gg 1$.

Landau level structure is typically only observable if one goes down to liquid Helium temperatures ~ 5°K.