To compute $\delta \rho$ we assume $V(\vec{r})$ is slowly varying so that the system is in local equilibrium at every position $\vec{r}$. This approx is good for getting the small $\vec{r}$ limit of $\delta \rho (\vec{r})$.

Then the probability to have an electron with wave vector $\vec{k}$ at position $\vec{r}$ is given by the Fermi function

$$f(\vec{k}, \vec{r}) = \frac{1}{e^{\frac{(\epsilon_k - eV_{\text{tot}}(\vec{r}) - \mu)}{k_B T}} + 1}$$

$$= f^0(\vec{k}; \mu + eV_{\text{tot}}(\vec{r}))$$

where $f^0(\vec{k}; \mu) = \frac{1}{e^{\frac{(\epsilon_k - \mu)}{k_B T}} + 1}$ is the equilibrium distribution when $V = 0$.

So the effect of $V_{\text{tot}}(\vec{r})$ can be viewed as if there is now a spatially varying chemical potential $\mu + eV_{\text{tot}}(\vec{r})$ (this is sometimes called the electro-chemical potential).

Then

$$\delta \rho(\vec{r}) = -e \int \frac{d^3k}{4\pi^3} \left[ f^0(\vec{k}; \mu + eV_{\text{tot}}(\vec{r})) - f^0(\vec{k}; \mu) \right]$$

includes factor $x^2$ for spin degeneracy.
expand in small $eV^{tot} \ll \mu \approx \varepsilon_F$

$$\delta \rho(\vec{r}) = -e \int \frac{d^3k}{4\pi^3} \frac{\partial f^0}{\partial \mu} e^{V^{tot}(\vec{r})}$$

$$= -e^2 V^{tot}(\vec{r}) \frac{\partial}{\partial \mu} \int \frac{d^3k}{4\pi^3} f^0(\vec{k})$$

regularized density $n(\mu)$ as function of chemical potential $\mu$.

$$\delta \rho(\vec{r}) \propto \mu$$

So also $\delta \rho(\vec{r}) = -e^2 V^{tot}(\vec{r}) \frac{\partial m}{\partial \mu}$

$$\frac{\delta \rho(\vec{r})}{V^{tot}(\vec{r})} = -e^2 \frac{\partial m}{\partial \mu}$$

So dielectric function is

$$\varepsilon(\vec{r}) = 1 - \frac{4\pi}{k^2} \frac{\delta \rho(\vec{r})}{V^{tot}(\vec{r})}$$

$$\varepsilon(\vec{r}) = 1 + \frac{\varepsilon_0}{k^2} \frac{\partial m}{\partial \mu}$$

This is called the Thomas-Fermi dielectric function, and it can be written in the form

$$\varepsilon(\vec{r}) = 1 + \frac{\varepsilon_0}{k^2} + \frac{\varepsilon^2}{k^2}$$
where $1/k_0$ is called the screening length.

Before considering the physical consequences of $\Sigma(\vec{p})$ as above, let's first compute $k_0$.

As $T \to 0$, $\mu \to E_F$, the Fermi energy. Now $E_F$ is defined by

$$M = \int_0^{E_F} d\varepsilon \; g(\varepsilon) \Rightarrow \frac{dM}{dE_F} = g(E_F)$$

\[\text{density of states}\]

For a free electron gas, $g(E_F) = \frac{3}{2} \frac{M}{E_F}$

So $k_0 = 4\pi e^2 g(E_F) = \frac{6\pi e^2 M}{E_F}$

We can compare this to what one would get using classical Boltzmann statistics for the electrons, instead of the quantum Fermi-Dirac statistics. Then, the probability distribution for an electron with momentum $\vec{p}$ would be

$$f(\vec{p}, \vec{r}) = C e^{-\left(\frac{E_k - eV(\vec{r})}{kT}\right)}$$

\[\text{normalization constant}\]
Expanded for small $eV^{tot}$

$$ f(k, \mathbf{r}) = c e^{-\frac{E_k}{k_B T}} \left[ 1 + \frac{eV^{tot}(\mathbf{r})}{k_B T} \right] $$

$$ = f^0(k) \left[ 1 + \frac{eV^{tot}(\mathbf{r})}{k_B T} \right] $$

$$ \delta f(\mathbf{r}) = -e \int \frac{d^3k}{4\pi^2} \left[ f(k, \mathbf{r}) - f^0(k) \right] $$

$$ = -e \int \frac{d^3k}{4\pi^2} f^0(k) \frac{eV^{tot}(\mathbf{r})}{k_B T} $$

$$ \geq -e^2 V^{tot}(\mathbf{r}) \frac{m}{k_B T} $$

So

$$ \frac{\delta f(\mathbf{r})}{V^{tot}(\mathbf{r})} = -\frac{e^2 m}{k_B T} \quad \text{which gives} $$

$$ \varepsilon(k) = 1 - \frac{4\pi e^2}{k^2} \frac{\delta f(\mathbf{r})}{V^{tot}(\mathbf{r})} = 1 + \frac{4\pi e^2}{k^2} \frac{m}{k_B T} $$

This is known as the Debye-Hückel dielectric function—it applies to a liquid or gas of charged particles obeying classical statistics (for example $T > T_F$). It has the same functional form as the Thomas-Fermi dielectric function, but now with

$$ k_0^2 = \frac{4\pi e^2 m}{k_B T} $$
So to compare Thomas-Fermi with classical Debye-Hückel

\[
\frac{k_0^{TF}}{k_0^{DH}} = \left( \frac{6\pi e^2 M}{k_B T_F} \frac{k_B T}{4\pi e^2 M} \right)^{1/2} \approx \left( \frac{3}{2} \left( \frac{T}{T_F} \right) \right)^{1/2} \ll 1
\]

so Debye-Hückel screening length \( \frac{1}{k_0^{DH}} \ll \frac{1}{k_0^{TF}} \) Thomas-Fermi screening length, for the same density \( n \).

Back to Thomas-Fermi

\[
k_0 = \frac{6\pi e^2 M}{E_F}
\]

using \( M = \frac{k_F^3}{3\pi^2} \) so \( E_F = \frac{\hbar^2 k_F^2}{2m} \)

\[
\Rightarrow k_0^2 = \frac{4\pi e^2}{\hbar^2} k_F
\]

using Bohr radius \( a_0 = \frac{\hbar^2}{me^2} \)

\[
\Rightarrow k_0^2 = \frac{4}{\pi} k_F a_0
\]

using \( \frac{4}{3} \pi r_s^3 = \frac{1}{M} \)

\[
k_F = \left( \frac{9\pi}{4} \right)^{1/3} r_s
\]

\[
\Rightarrow k_0 = \sqrt{\frac{4}{\pi}} \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{\sqrt{r_s a_0}}
\]

\[
\frac{\sqrt{\frac{9\pi}{4}}}{\sqrt{r_s a_0}} = 1.56
\]

\[
\frac{1}{a_0 k_0} = \frac{1}{1.56} \sqrt{r_s a_0}
\]

\( r_s \sim 3a_0 \) for most metals

\[
\Rightarrow \frac{1}{k_0} \sim a_0 \approx 0.5 \text{ Å}
\]

screening length very small!
Thomas-Fermi dielectric function

$$\varepsilon(k) = 1 + \frac{k_0}{k^2}$$, with $\frac{1}{k_0} \gg \AA$

One consequence of this form is something you have already learned in your EM class.

If we take the limit $k \to 0$, then $\varepsilon \to \infty$. So a uniform electric field applied to a metal is completely screened out! $E^{\text{ext}} = \frac{E}{\varepsilon}$.

In practice, provided the applied $E$ field is slowly varying on the length scale $\frac{1}{k_0} \gg \AA$, it is still screened out because $\varepsilon$ is so large for $k \ll k_0$. This is what you learned in EM - there can be no static macroscopic electric field inside a metal.

Another extremely important consequence of the T-F dielectric function is seen if we consider the effect on a point charge $Q$ placed in the electron gas.

The "applied" potential $V$ from the point charge is just the bare Coulomb potential

$$V(r) = \frac{Q}{r}$$
The Fourier transform of the Coulomb potential is

\[ V(\vec{r}) = \frac{4\pi e}{\varepsilon(\vec{r}) k^2} \]

\[ \Rightarrow V^{\text{tot}}(\vec{r}) = \frac{V(\vec{r})}{\varepsilon(\vec{r})} = \frac{4\pi e}{k^2} \frac{1}{1 + k_0^2/k^2} \]

\[ = \frac{4\pi e}{k_0^2 + k^2} \]

whereas \( V(\vec{r}) \to \infty \) as \( k \to 0 \), indicating a long-ranged interaction, \( V^{\text{tot}}(\vec{r}) \to \text{const} \) as \( k \to \infty \), indicating a short-ranged interaction.

If we Fourier transform \( V^{\text{tot}}(\vec{r}) \) back to real space, we get the interaction

\[ V^{\text{tot}}(\vec{r}) = \frac{4\pi e}{\varepsilon(\vec{r})} e^{-k_0 r} \]

called the "Yukawa potential" or the "screened Coulomb potential."

The effect of the dielectric function due to the free electrons is to "screen" the long-range Coulomb potential so it looks short ranged with an interaction length \( = 1/k_0 \). On length scales \( r \gg 1/k_0 \), the effect of the charge \( Q \) is entirely negligible. We say that the electrons have screened out the charge \( Q \).
Physically, what is going on is as follows:

Electrons get attracted to charge $\Phi$ and so the average electron density about $\Phi$ increases above average. The cloud of electron charge at $\Phi$ "seems" the charge $\Phi$. If one computes the total charge (i.e., $\Phi$ + induced electron charge) in a sphere of radius $r$ centered on $\Phi$, the total charge decreases to zero as $r \to \infty$. Decay of the total charge is on length scale $1/k_0$.

Compare this to behavior in a "dielectric" (i.e., an insulator) from EM class. If you put a point charge $\Phi$ in a dielectric, it polarizes the material creating bound charges at $\Phi$ so that the total charge at $\Phi$ becomes $\Phi + \text{bound} = \Phi/\varepsilon$, where $\varepsilon$ is the finite "dielectric constant". A metal is like a dielectric with an infinite dielectric constant so that $\Phi/\varepsilon \to 0$! (Any free charge in a metal must lie on its surface!). The dependence of $\varepsilon$ on wave vector $k$, described how the metal screens charges on finite length scales so $\Phi/\varepsilon \to 0$ is really just a statement about the $r \to \infty$, or $k \to 0$ limit of the metal.
about the screening of the electron-electron interaction. Since all electrons are identical, we cannot really distinguish between a given pair of interacting electrons and the other electrons that are screening this interaction. But despite this complication, the idea that E/w screening the e-e interaction and makes it short ranged, remains essentially correct. See Ashcroft & Mermin Chpt 17 or Kittel Chpt 14 for further discussion.