Plasmon

Although we argued by screening that e-e interactions are less important than one might naively expect, nevertheless the Coulomb interaction between electrons does give rise to physically interesting effects.

One such effect is the plasmon - which is a longitudinal charge density oscillation.

**Simple explanation:** consider the gas of electrons as a rigid charged body of mass \( mn = mN \) where \( N \) is the total number of electrons. If we displace the electrons a distance \( d \) with respect to the ions, we will create a surface charge on the surfaces of the system as shown below.

\[
\begin{align*}
+\sigma &= \text{electrons} \\
-\sigma &= \text{ions}
\end{align*}
\]

Surface charge \( \sigma \) creates electric field inside

\[
\vec{E} = 4\pi \sigma \hat{x} = 4\pi m\sigma \hat{x}
\]
Newton's equation of motion for the electrons is then
\[ mN \ddot{d} = -eNE = -4\pi me^2 dN \]

\[ \ddot{d} = \frac{-4\pi me^2}{m} d \]

\( \Rightarrow \) harmonic oscillation at frequency \( \omega_p = \sqrt{\frac{4\pi me^2}{m}} \)

oscillation in charge and \( \vec{E} \) with freq \( \omega_p \).

Another way to set plasma oscillations from Maxwell's equations

When we considered EM wave propagation in a metal early in the course, we limited discussion to transverse modes where \( \vec{k} \cdot \vec{E} = 0 \). The plasma oscillation is a longitudinal mode \( \vec{k} \cdot \vec{E} \neq 0 \).

Charge conservation: \( \nabla \cdot \vec{f} = -\frac{\partial \vec{f}}{\partial t} \)

for harmonic oscillation: \( \vec{f} = f_0 e^{-i\omega t} e^{i\vec{k} \cdot \vec{r}} \)

freq \( \omega \), wavevector \( \vec{k} \) \( f = f_0 e^{-i\omega t} e^{i\vec{k} \cdot \vec{r}} \)

\( \Rightarrow \) \( i\vec{k} \cdot \vec{f}_0 = i\omega f_0 \)

But we also had \( \vec{f}_0 = \alpha(\omega)\vec{E}_0 \) \( \alpha \) in conductivity

\( \Rightarrow \) \( i\vec{k} \cdot \alpha \vec{E}_0 = i\omega f_0 \)
From Gauss's Law \[ \nabla \cdot \vec{E} = 4\pi \rho \]

\[ \Rightarrow \quad i \vec{k} \cdot \vec{E}_0 = 4\pi \rho \]

Combine above with charge conservation to get

\[ \frac{\sigma}{\varepsilon_0} \vec{k} \cdot \vec{E}_0 = i \frac{\vec{k} \cdot \vec{E}_0}{4\pi} \]

\[ \Rightarrow \quad \vec{k} \cdot \vec{E}_0 = 0 \Rightarrow \text{transverse mode} \]

or

\[ \Rightarrow \quad \frac{4\pi \sigma}{\varepsilon_0} = 1 \]

\[ \Rightarrow \quad 1 + \frac{4\pi i \sigma}{\varepsilon_0} = 0 \]

We saw the above quantity earlier in our discussion of transverse wave propagation in metals. Then we had for the dispersion relation for the transverse EM waves:

\[ k^2 = \frac{\omega^2}{c^2} \left[ 1 + \frac{4\pi i \sigma}{\varepsilon_0} \right] \]

In analogy with dielectrics, one sometimes defines complex electric function

\[ \varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\varepsilon_0} \quad \text{for a metal} \]

complex dielectric frequency dependent electric function
Longitudinal oscillations occur when

\[ \varepsilon(\omega) = 1 + \frac{4\pi e^2}{\omega} \frac{\Im(\varepsilon)}{\varepsilon} = 0 \]

From our discussion of the Drude model, we had

\[ \sigma(\omega) = \sigma_{dc} \frac{\omega}{1 - i\omega \tau} \]

\[ \sigma_{dc} = \frac{me^2}{\hbar} \]

For high frequencies \( \omega \tau \ll 1 \), \( \sigma(\omega) \approx \sigma_{dc} \frac{\omega}{i\omega \tau} \)

and so

\[ \varepsilon(\omega) = 1 - \frac{4\pi me^2}{m \omega^2} \]

\[ = 1 - \left( \frac{\omega_p}{\omega} \right)^2 \quad \text{with} \quad \omega_p = \sqrt{\frac{4\pi me^2}{m}} \]

So the condition \( \varepsilon(\omega) = 0 \) for longitudinal modes of oscillation

\[ \Rightarrow \left\{ \omega = \omega_p \right\} \quad \text{for any wavevectors} \quad \hat{k} \]

Such longitudinal modes are called "plasma" oscillations since they are accompanied by the longitudinal oscillations of the electric field \( \vec{k} \cdot \vec{E}_0 \neq 0 \) are (by Gauss' law) accompanied by oscillations in electron charge density.
Note, the above Maxwell argument gives a plasma oscillation at \( \omega = \omega_p \) for any longitudinal wave vector \( \mathbf{k} \). In reality, the plasma frequency of plasma oscillations does depend on \( \mathbf{k} \).

In our derivation of \( \tilde{\varepsilon}(\omega) \) we assumed that the wavelength \( \lambda \) of the EM oscillations was macroscopically large, \( \lambda > > \) atomic lengths. This leads to a \( \tilde{\varepsilon}(\omega) \) independent of wavevector \( \mathbf{k} \), (i.e., we ignored spatial dependence of \( \tilde{\varepsilon} \) on equation of motion of electron). When one does a better job, one finds that \( \varepsilon = 1 + \frac{4\pi \varepsilon_0 \omega_p^2}{\omega} \) should really have a dependence on \( \mathbf{k} \) as well, that is important when \( k \) is of the order \( \frac{1}{\lambda} \), i.e., \( \nabla \alpha \), atomic length scale. (Recall the \( k \)-dependence of the Thomas-Fermi dielectric function \( \varepsilon_k \) for the \( \omega = 0 \) case). If one includes this \( k \)-dependence of \( \varepsilon(\mathbf{k}, \omega) \), then the condition \( \varepsilon(\mathbf{k}, \omega) = 0 \) gives a dispersion relation for plasma oscillations:

\[
\omega_p(\mathbf{k}) = \omega_p \left[ 1 + \frac{3}{10} \frac{V_F^2 k^2}{\omega_p^2} \right]
\]

where \( \omega_p = \sqrt{\frac{4\pi e^2}{m}} \) as before and \( V_F \) is the Fermi velocity.

Note \( \frac{V_F^2 k^2}{\omega_p^2} = 4 \left( \frac{\varepsilon_F}{k_{\text{F}}} \right)^2 \left( \frac{k}{k_{\text{F}}} \right)^2 \).
For typical metals, $E_F \sim 2-10$ eV

$\hbar w_p \sim 10-20$ eV

⇒ correction to $w_p$ at finite $k$ is usually quite small for $k < k_F$.

As with other harmonic oscillations, the longitudinal plasma oscillations of electrons in a metal, get quantized in a more complete quantum mechanical treatment of the EM fields. When so quantized, the plasma oscillations are referred to as "plasmons".

The energy associated with the $n$th level of excitation of the oscillations with wave vector $\vec{k}$, i.e., the energy of $n$ plasmons of wave vector $\vec{k}$, is just $(n+\frac{1}{2}) \hbar w_p(\vec{k})$.

Because $\hbar w_p \sim 10-20$ eV $\gg k_B T$, plasmons are not in general thermally excited. However, the zero point energy of the plasmon modes, $\frac{1}{2} \hbar w_p(\vec{k})$, does contribute to the total ground state energy of the electron gas.

When one shoots a high energy electron into a metal surface, one can see energy losses corresponding to the excitation of integer numbers of plasmons with energies $n \hbar w_p$. 
Another moral from the story of the plasmon:

We start with electrons, which are fermions. A *bare* electron has energy $\varepsilon(k) = \frac{p^2}{2m}$. When we include effects of the Coulomb interactions among the electrons in a gas of electrons, we get not only fermionic degrees of freedom with dispersion relation $\varepsilon(k) = \frac{p^2}{2m}$, but now we also get bosonic degrees of freedom, i.e. the plasmons, with dispersion relation

$$k \omega_p(k) \approx k \omega_p \left(1 + \frac{3}{10} \frac{V^2 k^2}{\omega_p^2} \right)$$

(often constant $\omega_p$ as $k \to 0$; weak dependence on $k$ for small $k < k_F$).

**Moral:** The presence of strong interactions among the "bare" (i.e. isolated) degrees of freedom can lead to elementary excitations (i.e. new degrees of freedom) of the system that bear no resemblance at all to the bare degrees of freedom — i.e. they can have a completely different dispersion relation $\varepsilon(k)$ and can even have different symmetry, i.e. bosonic instead of fermionic. This is a general rule to remember in all fields of physics! (Another condensed matter example is phonons: bare ions have $\varepsilon(k) = \frac{k^2}{2M}$, but the interacting ions lead to quantized elastic vibrations (phonons) with $k \omega(k) \sim C + k - \text{sound modes}$).
Although we argued that e-e interactions are screened and so less important than one might expect, Wigner argued that the free-electron-like filled Fermi sphere ground state could become unstable to an insulating lattice of localized electrons, when the density of the electron gas gets sufficiently small. The formation of this Wigner electron crystal was proposed to be due to a competition between electrostatic potential energy and electron kinetic energy.

Wigner's argument applies to a homogeneous electron gas with a fixed uniform neutralizing background of positive charge (i.e. instead of point positive ions). A simple argument is as follows.

Consider the electrons localized to the points of a periodic lattice of sites. Each electron occupies a volume. The volume per electron is \( V = \frac{V}{N} \).

We can imagine dividing the space up into spheres of radius \( R_s \) \( \left( \frac{4}{3} \pi R_s^3 = V \right) \) with uniform positive charge filling the space at the center of the sphere. Of course such spheres may slightly overlap, and leave some voids in the regions where they meet. Neighboring spheres meet, but we ignore such complications for the sake of simplicity. Since each sphere is
neutral, Gauss law gives that the field outside each sphere will vanish, hence these spheres have little or no interaction between them. The electrostatic energy per electron is then just the electrostatic energy of the electron and its uniform sphere of positive charge. On dimensional grounds we can estimate this energy as $-\frac{e^2}{4\pi\epsilon_0 r_s}$. Or we can do a calculation as follows:

Total electrostatic energy has two pieces

$$U = U_{\text{ep}} + U_{\text{pp}}$$

Where $U_{\text{ep}}$ is interaction of electron with positive charge and $U_{\text{pp}}$ is interaction of positive charge with itself.

We can get both by computing the electrostatic potential $V(r)$ due to the uniform sphere of positive charge.

$$\text{charge density } \sigma = \frac{e}{\frac{4}{3}\pi r_s^3}$$

From Gauss law, $E$ is radially symmetric and in radial direction. Gauss law then gives for surface of radius $r$:

$$\int E^2 \, d\mathbf{a} = 4\pi r^2 E(r) = 4\pi \rho \frac{4}{3}\pi r^3 \quad r < r_s$$

$$E(r) = \begin{cases} \frac{\epsilon_0}{r} & r > r_s \\ \frac{\sigma}{\epsilon_0} & r < r_s \end{cases}$$
Substitute for \( f \)

\[
E(r) = \begin{cases} 
\frac{4}{3} \pi \rho r = \frac{e r}{r_s^3} & r < r_s \\
\frac{e}{r^2} & r > r_s
\end{cases}
\]

\[
-\frac{dV}{dr} = E \Rightarrow V(r) = \begin{cases} 
-\frac{e r^2}{2 r_s^3} + \text{const} & r < r_s \\
\frac{e}{r} & r > r_s
\end{cases}
\]

\( V \) continuous at \( r = r_s \) \( \Rightarrow \) \( \text{const} - \frac{e}{2 r_s^3} = \frac{e}{r_s^3} \)

\( \text{const} = \frac{3}{2} \frac{e}{r_s^3} \)

\[
V(r) = \begin{cases} 
\frac{e}{2 r_s^3} \left\{ 3 - \frac{r^2}{r_s^2} \right\} & r < r_s \\
\frac{e}{r} & r > r_s
\end{cases}
\]

Self energy of positive charge \( \nu \)

\[
U_{\nu \nu} = \frac{1}{2} \int d^3r \rho V = \frac{4 \pi}{2} \rho \left( \frac{4 \pi}{3} \int_0^{r_s} dr r^2 V(r) \right)
\]

\[
= \frac{4 \pi}{2} \frac{e}{4 \pi r_s^3} \int_0^{r_s} dr \frac{e}{2 r_s} \left\{ 3 r^2 - \frac{r^4}{r_s^2} \right\}
\]

\[
= \frac{3}{2} \frac{e^2}{r_s^4} \left( r_s^3 - \frac{r_s^5}{5 r_s^2} \right) = \frac{3}{4} \frac{e^2}{r_s^4} r_s^3 \frac{4}{5}
\]

\[
U_{\nu \nu} = \frac{2}{5} \frac{e^2}{r_s^3}
\]
energy of electron - positive charge interaction is

\[ U_{\text{ep}} = -e V(0) = -\frac{e^2}{2r_s^2} \]

\[ U = U_{\text{ep}} + U_{\text{pp}} = -\frac{e^2}{r_s^2} \left( \frac{3}{2} - \frac{3}{5} \right) = -\frac{e^2}{r_s^2} \left( \frac{15 - 6}{10} \right) \]

\[ U = -\frac{9}{10} \frac{e^2}{r_s^2} \]

↑ total electrostatic energy per electron of Wigner electron lattice.

We now have to add on the kinetic energy of the electron confined to the sphere of radius \( r_s \).

A naive estimate of kinetic energy is as follows:

For an electron in a sphere of radius \( r_s \), the wave function is \( \psi \sim r \)\(^2 \) \( r_s \),

\[ k = \frac{2\pi}{r_s} \]

\[ \Rightarrow \text{kinetic energy} = \frac{\hbar^2}{2m} \sim \frac{4\pi^2 \hbar^2}{2m r_s^2} \]

Total energy per electron of Wigner lattice is

\[ E_W = -\frac{9}{10} \frac{e^2}{r_s^2} + 4\pi^2 \frac{\hbar^2}{2m r_s^2} \]

Compare this to the energy per electron of the filled Fermi sphere

\[ E_F = \frac{3}{5} \varepsilon_F \]
To compare these two energies:

\[ E_W = -\frac{9}{10} \frac{e^2}{a_0} \left( \frac{a_0}{r_S} \right) + \frac{4\pi^2}{2} \frac{\hbar^2}{m_e} \frac{e^2}{r_S^2} \]

\[ \text{use } a_0 = \frac{\hbar^2}{m_e} \]

\[ E_W = -\frac{9}{10} \frac{e^2}{a_0} \left( \frac{a_0}{r_S} \right) + 2\pi^2 \frac{e^2}{a_0} \left( \frac{a_0}{r_S} \right)^2 \]

\[ = \frac{e^2}{a_0} \left[ -\frac{9}{10} \left( \frac{a_0}{r_S} \right) + 2\pi^2 \left( \frac{a_0}{r_S} \right)^2 \right] \]

whereas

\[ \sqrt{\text{from lecture 4}} \]

\[ E_F = \frac{3}{5} E_F = \frac{3}{5} \frac{e^2}{2a_0} \left( kF a_0 \right)^2 = \frac{3}{10} \frac{e^2}{a_0} \left( 1 - \frac{1}{2} \right)^2 \left( \frac{a_0}{r_S} \right)^2 \]

So the energy difference is:

\[ E_W - E_F = -\frac{e^2}{a_0} \left\{ -\frac{9}{10} \left( \frac{a_0}{r_S} \right) + 2\pi^2 \left( \frac{a_0}{r_S} \right)^2 \right. \]

\[ \left. + \frac{6}{5} \left( \frac{a_0}{r_S} \right)^2 \right\} \]

\[ = -\frac{e^2}{a_0} \left\{ -\frac{9}{10} - 18 \left( \frac{a_0}{r_S} \right) \right\} \left( \frac{a_0}{r_S} \right) \]

So the Wigner lattice will have lower energy than the filled Fermi sphere (and hence will be the better ground state) when

\[ E_W - E_F < 0 \Rightarrow \frac{9}{10} - 18 \left( \frac{a_0}{r_S} \right) > 0 \]

\[ \Rightarrow r_S > 20 a_0 \]
So for sufficiently dilute electron gas, the Wigner lattice should become the ground state because the negative electrostatic energy outweighs the increase in kinetic energy.

The above was a rough calculation. Clearly our estimate for both potential and kinetic energy terms for the Wigner lattice were rough estimates.

A more advanced calculation, using density functional method (Ceperley + Alder, PRL 45, 566 (1980)) gives the critical value of \( r_s \) as

\[ r_s \approx 100 a_0 \]