

Cooper pairing

An arbitrary weak but attractive interaction between ^{two} electrons excited above the filled Fermi surface leads to a bound state of the electrons with energy $E < 2\varepsilon_F$. This then leads to an instability of the filled Fermi sphere to such ~~pair~~ bound pair formation, that completely changes the nature of the ground state of the N -electron system and leads to the phenomenon of superconductivity (BCS - Bardeen-Cooper-Schrieffer theory of superconductivity). The presence of the filled Fermi sphere is crucial to the effect - compare to two isolated particles in 3D where a bound state will not form unless the interaction exceeds a certain strength.

Consider a pair of electrons excited above the Fermi surface ε_F . Assume that the ground state of this pair will have zero net momentum and zero net spin (singlet spin state). (Since the interaction is attractive \Rightarrow ~~most fav~~ electrons prefer to be near each other \Rightarrow most favorable wavefunction is spatially symmetric, so it must be antisymmetric in spin).

Let \vec{r}_1 and \vec{r}_2 be the positions of the two electrons.
 Assume that the two-particle wavefunction has the form:

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{V} \sum_{\vec{k}} g_{\vec{k}} e^{i\vec{k} \cdot \vec{r}_1} e^{-i\vec{k} \cdot \vec{r}_2} \quad (V \text{ is volume})$$

i.e. $\vec{k}_1 = -\vec{k}_2$ so that total momentum of the pair is zero.

Since the electrons are above a filled Fermi sphere we must have $g_{\vec{k}} = 0$ for all $|\vec{k}| < k_F$ since these states are already occupied.

If $U(|\vec{r}_1 - \vec{r}_2|)$ is the interaction between the two electrons, then the Schrödinger equation is

$$-\frac{\hbar^2}{2m} [\nabla_1^2 + \nabla_2^2] \psi + U(|\vec{r}_1 - \vec{r}_2|) \psi = E \psi$$

Use Fourier transform $U(|\vec{r}_1 - \vec{r}_2|) = \frac{1}{V} \sum_{\vec{q}} U_{\vec{q}} e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)}$

Plug into Schrödinger equation to get:

$$\frac{1}{V} \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} [k^2 + k^2] g_{\vec{k}} e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} + \frac{1}{V} \sum_{\vec{k}', \vec{q}} U_{\vec{q}} e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)} g_{\vec{k}'} e^{i\vec{k}' \cdot (\vec{r}_1 - \vec{r}_2)}$$

$$= E \frac{1}{V} \sum_{\vec{k}} g_{\vec{k}} e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)}$$

$$\Rightarrow \sum_{\vec{k}} \left\{ \frac{\hbar^2 k^2}{m} g_{\vec{k}} + \frac{1}{V} \sum_{\vec{k}'} U_{\vec{k}-\vec{k}'} g_{\vec{k}'} - E g_{\vec{k}} \right\} e^{i\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} = 0$$

where we made substitution $\vec{q} = \vec{k} - \vec{k}'$ in the potential term

$$\Rightarrow \frac{\hbar^2 k^2}{m} g_k + \frac{1}{V} \sum_{k'} U_{k-k'} g_{k'} = E g_k \quad \left. \begin{array}{l} \text{Bethe} \\ \text{- Goldstone} \\ \text{equation} \end{array} \right\}$$

$$g_k = 0 \quad \text{for } |\vec{k}| < k_F$$

using $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ we have

$$(E - 2\epsilon_k) g_k = \frac{1}{V} \sum_{k'} U_{k-k'} g_{k'}$$

This is very difficult to solve for a general $U_{k-k'}$.
To simplify, we make a crude approximation:

$$U_{k-k'} = \begin{cases} -U_0 & \text{if } \epsilon_k, \epsilon_{k'} \text{ within } \overbrace{\hbar\omega_D}^{\text{max phonon energy}} \text{ of } \epsilon_F \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow g_k = \frac{-U_0 \left(\frac{1}{V} \sum_{k'} g_{k'} \right)}{E - 2\epsilon_k} \quad \begin{array}{l} \text{where } \Sigma' \text{ means a} \\ \text{sum over } \vec{k}' \text{ such} \\ \text{that } |\vec{k}'| > k_F \text{ and} \\ \frac{\hbar^2 k'^2}{2m} < \epsilon_F + \hbar\omega_D \end{array}$$

Now sum both sides over \vec{k}

$$\left(\sum_k g_k \right) = -U_0 \left(\frac{1}{V} \sum_{k'} g_{k'} \right) \left(\sum_k \frac{1}{E - 2\epsilon_k} \right)$$

cancel $\sum g_k$ from both sides to get

$$1 = -U_0 \frac{1}{V} \sum_k' \frac{1}{E - 2\varepsilon_k}$$

$$\Rightarrow 1 = -U_0 \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} d\varepsilon \frac{g(\varepsilon)}{E - 2\varepsilon} \quad g(\varepsilon) \text{ is density of states.}$$

$$\approx -U_0 g(\varepsilon_F) \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} d\varepsilon \frac{1}{E - 2\varepsilon}$$

where we assumed $g(\varepsilon)$ varied slowly from ε_F to $\varepsilon_F + \hbar\omega_D$ (true since $\hbar\omega_D \ll \varepsilon_F$)

$$\Rightarrow 1 = \frac{U_0 g(\varepsilon_F)}{2} \ln \left(\frac{2\varepsilon_F - E + 2\hbar\omega_D}{2\varepsilon_F - E} \right)$$

solve above for the energy E

$$E = 2\varepsilon_F - \frac{2\hbar\omega_D}{e^{2/g(\varepsilon_F)U_0} - 1}$$

For a weak potential, $U_0 g(\varepsilon_F) \ll 1$, we have

$$E - 2\varepsilon_F = -2\hbar\omega_D e^{-2/g(\varepsilon_F)U_0}$$

Since the pair of electrons, in the absence of the attractive potential U , would have a minimum energy of $2\varepsilon_F$, the binding energy of the pair is

$$E' = 2\varepsilon_F - E = 2\hbar\omega_D e^{-2/g(\varepsilon_F)U_0} > 0$$

\Rightarrow bound state E has a lower energy than $2\varepsilon_F$

Note that the binding energy $\sim e^{-2/g(E_F)U_0}$ is a non-analytic function of U_0 , i.e. it cannot be expanded in powers of U_0 . This means that we could never have gotten this result by using perturbation theory!

Conclusion: pairs of electrons at the Fermi surface E_F can lower their energy by binding together into such a "Cooper pair". \Rightarrow The filled Fermi sphere ~~can no longer~~ is unstable to the formation of Cooper pairs and so can no longer be the true ground state. The new ground state was obtained by BCS starting from this idea of Cooper pairs, and became the basis for understanding superconductivity.

Back to the Cooper pair wavefunction:

$$g_{\vec{k}} \sim \frac{U_0 \times \text{constant}}{2E_{\vec{k}} - E} \quad \begin{array}{l} \text{depends on } \vec{k} \\ \text{only via } E_{\vec{k}} \end{array}$$

\Rightarrow Solution is spherically symmetric \Rightarrow Cooper pairs bind in an s-wave state.

The maximum $g_{\vec{k}}$ occurs for the smallest $E_{\vec{k}}$, i.e. at $|\vec{k}| = k_F$. Using $E = 2E_F - E'$ we get

$$g_k \sim \frac{U_0}{2(\epsilon_k - \epsilon_F) + E'}$$

The distribution g_k as function of k has a width given by

$$\epsilon_k - \epsilon_F \approx E' \ll \hbar \omega_D$$

Since only states within E' of ϵ_F are important in making the bound state, and $E' \ll \hbar \omega_D$, this suggests that the detailed structure of U_k is not so crucial and so our approximation might not be so terrible.

The size of a Cooper pair is ~~greater than~~ can be estimated as follows:

$$\xi_0 \sim \frac{\hbar v_F}{E'}$$

$$\xi_0 \sim \Delta R \sim \frac{1}{\Delta k} = \frac{1}{\frac{dR}{dE} \Delta E} = \frac{dE}{dR} \frac{1}{\Delta E} = \frac{\hbar v_F}{E'}$$

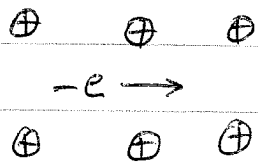
where we used that width in k -space was determined by $\epsilon_k - \epsilon_F \approx E'$.

$$\xi_0 \sim \frac{\hbar v_F}{E'} \gg r_s \quad \left(\frac{4\pi r_s^3}{3} = \frac{1}{n} \right)$$

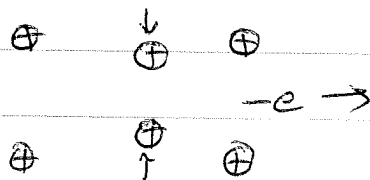
So the spatial extent of a Cooper pair is very much larger than the spacing between electrons

What is the origin of the attractive $e-e$ interaction that leads to Cooper pairing?

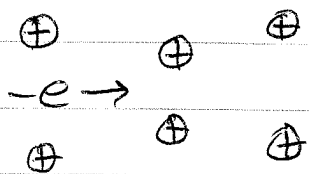
It is a time-delayed ionic screening effect!



electron passes by



ions get attracted to electron and deform. Net (+) charge builds up where $-e$ passed by, but now the $-e$ electron has moved on!



The excess (+) charge where the ions have deformed then attracts a new electron to the place where the first electron had been. This leads to an effective (but time delayed) attractive correlation between the electrons.

Crucial to this picture is that the ions move much more slowly than the electrons, so the ^{ion} deformation that attracts the 2nd electron, remains after the 1st electron has past. That is how the two (-) electrons can still attract!