

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

The real space orbits $(\vec{r}(t))$ can be found:

$$\vec{r}_\perp \equiv \vec{r} - \hat{H}(\hat{H} \cdot \vec{r}) \quad \text{position in plane } \perp \text{ to } H$$

$$\begin{aligned} \hat{H} \times \hbar \dot{\vec{k}} &= -\frac{e\hbar}{c} \hat{H} \times (\vec{v} \times \hat{H}) = -\frac{e}{c} H (\vec{r} - \hat{H}(\hat{H} \cdot \vec{r})) \\ &= -\frac{eH}{c} \vec{r}_\perp \end{aligned} \quad \begin{array}{l} \text{using } \vec{v} = \dot{\vec{r}} \\ \text{\& vector identity} \end{array}$$

$$\text{so } \vec{r}_\perp(t) - \vec{r}_\perp(0) = -\frac{\hbar c}{eH} \hat{H} \times (\vec{k}(t) - \vec{k}(0))$$

so \vec{r}_\perp orbit is just \vec{k} orbit rotated by 90° about \hat{H} , and scaled by $\frac{\hbar c}{eH}$

in || direction

$$r_{||}(t) = r_{||}(0) + \int_0^t v_{||}(t) dt = r_{||}(0) + \int_0^t \frac{1}{\hbar} \frac{\partial \epsilon}{\partial k_{||}} dt$$

need not be uniform in t as $\frac{\partial \epsilon}{\partial k_{||}}(\vec{k})$ can vary as k_\perp varies.

For spherical energy surface, we get classical result: electron moves in circular orbit \perp to \hat{H} .

However energy surfaces need not be spherical - (when they get too near zone boundaries) - need not be closed curves! See figure 12.8 in text

~~When orbits are open, applying H can lead to~~

in nearly free electron model

$$E(\vec{k}) \approx \frac{\hbar^2 k^2}{2m}$$

surface of constant energy E
is sphere of radius

$$\sqrt{\frac{2mE}{\hbar^2}} = k \quad \text{in } k\text{-space}$$

$$\bar{E}(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} - \hbar \vec{w} \cdot \vec{k} \quad \text{surface of constant } \bar{E}$$

is given by

$$\frac{\hbar^2}{2m} \left| \vec{k} - \frac{m\vec{w}}{\hbar} \right|^2 = \bar{E} + \frac{1}{2}m\vec{w}^2$$

sphere in k -space of radius

$$k = \sqrt{\frac{2m}{\hbar^2} \left(\bar{E} + \frac{1}{2}m\vec{w}^2 \right)}$$

centered about $\vec{k}_0 = m\vec{w}/\hbar$

surface of constant \bar{E} is
shifted by $\vec{w} \cdot \vec{k}$ term in direction
 \vec{w}

Motion in uniform \perp \vec{E} and \vec{H} fields
Hall effect and magnetoresistance

$$\hbar \dot{\vec{k}} = -e \left[\vec{E} + \frac{\vec{v}}{c}(\vec{k}) \times \vec{H} \right]$$

$$\Rightarrow \hat{H} \times \hbar \dot{\vec{k}} = -e \hat{H} \times \vec{E} - \frac{eH}{c} \dot{\vec{r}}_{\perp}$$

$$\dot{\vec{r}}_{\perp} = -\frac{\hbar c}{eH} \hat{H} \times \dot{\vec{k}} + \vec{w} \quad \vec{w} = \frac{cE}{H} (\hat{E} \times \hat{H})$$

Motion is as before, but with drift velocity \vec{w} added.

To determine orbits in k space note:

$$\hbar \dot{\vec{k}} = -e \vec{E} - \frac{e}{c} \frac{\perp}{\hbar} \frac{\partial \epsilon}{\partial \vec{k}} \times \vec{H}$$

write $\vec{E} = -(\vec{E} \times \hat{H}) \times \hat{H}$
true when $\vec{E} \perp \vec{H}$

$$= -\frac{e}{c\hbar} \left(\frac{\partial \epsilon}{\partial \vec{k}} - \frac{c\hbar E}{H} \hat{E} \times \hat{H} \right) \times \vec{H}$$

$$\equiv -\frac{e}{c\hbar} \frac{\partial \bar{\epsilon}}{\partial \vec{k}} \times \vec{H} \quad \bar{\epsilon} = \epsilon - \hbar \vec{k} \cdot \vec{w}$$

Same as if \vec{E} was absent and band structure replaced by $\bar{\epsilon}(\vec{k}) = \epsilon(\vec{k}) - \hbar \vec{k} \cdot \vec{w}$

Orbits are intersections of surfaces of constant $\bar{\epsilon}$ with planes \perp to \vec{H}

We will assume that $-\hbar \vec{k} \cdot \vec{w}$ small enough so that if the constant $\epsilon(\vec{k})$ surface is closed (open) so is the constant $\bar{\epsilon}(\vec{k})$ surface. Good approx in most cases - see text for estimate of numbers.

Hall effect: $\dot{\vec{r}}_{\perp} = -\frac{\hbar c}{eH} \hat{H} \times \dot{\vec{k}} + \vec{w}$, $\vec{w} = \frac{cE}{H} (\hat{E} \times \hat{H})$

current in plane \perp to H is

$\vec{j} = -ne\langle \dot{\vec{r}}_{\perp} \rangle$ where $\langle \dot{\vec{r}}_{\perp} \rangle$ is steady state average over all occupied electron orbits, and over collisions

$\vec{j} = -ne\vec{w} + \frac{ne\hbar c}{eH} \hat{H} \times \langle \dot{\vec{k}} \rangle$

Case (1) All occupied (or unoccupied) orbits are closed. Then for large enough H so that $\omega_c \tau \gg 1$ (where τ is collision time, and $\omega_c = eH/m^*c$), electron makes many periods of its closed orbits between successive collisions.

We can estimate $\langle \dot{\vec{k}} \rangle$ in this large H case as follows: Averaging over electron motion between two successive collisions at $t=0$ and $t=t_0$ we get

$$\langle \dot{\vec{k}} \rangle = \frac{1}{t_0} \int_0^{t_0} \dot{\vec{k}}(t) dt = \frac{\vec{k}(t_0) - \vec{k}(0)}{t_0}$$

where $\vec{k}(0)$ is wave vector of electron as it emerges from the first collision at $t=0$, and $\vec{k}(t_0)$ is wave vector of electron just before second collision at $t=t_0$

As in the Drude model, we may assume that electrons emerge from a collision with an equilibrium distribution determined by the local temperature + chemical potential. Since the Fermi distribution $f(\epsilon(\vec{k})) = \frac{1}{1 + e^{\beta(\epsilon(\vec{k}) - \mu)}}$ depends on \vec{k} only thru energy $\epsilon(\vec{k})$, and $\epsilon(\vec{k}) = \epsilon(-\vec{k})$.

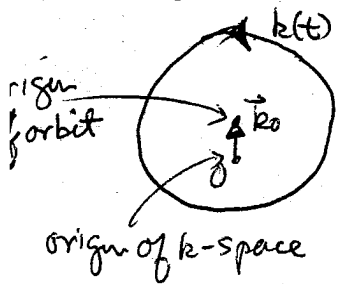
We have, after averaging over the electron emerging from the collision at $t=0$, $\langle \vec{k}(0) \rangle = 0$. So $\langle \vec{k} \rangle = \vec{k}(t_0)/t_0$.

We now average over the time until the second collision, $\langle t_0 \rangle = \tau$ (this time is distributed randomly with average equal to τ). Since $\omega_c \tau \gg 1$, the electron makes many orbits between collisions, $\Rightarrow \vec{k}(t_0)$ when averaged over collision time t_0 , is equally likely to lie anywhere along the closed orbit.

$\Rightarrow \langle \vec{k}(t_0) \rangle = (\text{average } \vec{k} \text{ on orbit})$. If electric field $\vec{E} = 0$, then (average \vec{k} on orbit) $= 0$ also. But when $E \neq 0$, (average \vec{k} on orbit) $\sim m^* \vec{w} / \hbar$. To see this, use effective mass approximation, $\epsilon(\vec{k}) \approx \frac{\hbar^2 k^2}{2m^*}$.

Then orbit lies on curve of constant

$\bar{\epsilon}(\vec{k}) = \epsilon(\vec{k}) - \hbar \vec{k} \cdot \vec{w}$, which lies on sphere centered at $\vec{k}_0 = m^* \vec{w} / \hbar$. So (average \vec{k} on orbit) $= \langle \vec{k}(t_0) \rangle = \vec{k}_0$



$$\Rightarrow \langle \dot{\vec{k}} \rangle = \frac{\langle \vec{k}(t_0) \rangle}{\tau} = \frac{\vec{k}_0}{\tau} = \frac{m^* \vec{w}}{\hbar \tau}$$

So contribution of $\langle \dot{\vec{k}} \rangle$ term to current is

$$\frac{ne \hbar c}{e \hbar} \hat{H} \times \frac{m^* \vec{w}}{\hbar \tau} = \frac{ne}{\omega_c \tau} \hat{H} \times \vec{w}$$

smaller than drift contribution to current

$$\vec{j} \approx -ne \vec{w} \quad \text{by a factor } \frac{1}{\omega_c \tau} \ll 1$$

So $\vec{j} \approx -ne \vec{w}$ given just by drift velocity \vec{w} in high field limit.

In this case \vec{j} is \parallel to \vec{w} $\Rightarrow \vec{j}$ is \perp to \vec{E} and \vec{H}
 \Rightarrow Lorentz force so strong that electrons move \perp to E and do not acquire any energy from the E -field.

The Hall coefficient in this limit is just $\frac{E_{\perp}}{jH}$ (and \perp to \vec{H})
 E -field perpendicular to H , but in this large H limit, this is just total E .

$$R_{H \rightarrow \infty} = \frac{E_{\perp}}{jH}$$

$$= \frac{E}{-nevH} \quad \text{but } \vec{w} = \frac{cE}{H} (\vec{E} \times \hat{H})$$

$$\Rightarrow R_{H \rightarrow \infty} = \frac{E}{\frac{-ne c E}{H} H} = \frac{-1}{nec}$$

Drude value

The above was for closed occupied orbits
If we had closed unoccupied orbits we would use the hole picture to get

$$R_{H \rightarrow \infty} = +\frac{1}{n_h ec} > 0$$

(n_h is density of holes, each hole has charge $+e$)

If there is more than one partially full band with only closed occupied or unoccupied orbits, then

$$\vec{j} = -n_{eff} \frac{ec}{H} (\vec{E} \times \hat{H}) \quad \text{where } n_{eff} = n - n_h$$

$$R_{H \rightarrow \infty} = \frac{-1}{n_{eff} ec}$$

total electron density in all partially full bands \uparrow
total hole density in partial full bands \uparrow

The effects of holes explains why R_{∞} can have non Drude values, and even be > 0 .

See text for what happens when $m_{\text{eff}} = 0$. This is case for undoped semiconductor

Another way to view things is ~~not~~ in terms of conductivity tensor, keeping contribution to \vec{j} from the $\langle \vec{k} \rangle$ term gives

$$\vec{j} = -ne\vec{w} + \frac{ne}{\omega_c\tau} \hat{H} \times \vec{w}, \quad \vec{w} = \frac{cE}{H} (\hat{z} \times \hat{H})$$

for $\hat{H} = \hat{z}$ direction we have

$$\vec{j} = \frac{ne c}{H} (\hat{z} \times \vec{E} + \frac{1}{\omega_c\tau} \vec{E}) = \vec{\sigma} \cdot \vec{E}$$

$$\text{with } \vec{\sigma} = \frac{ne c}{H} \begin{pmatrix} \frac{1}{\omega_c\tau} & -1 \\ 1 & \frac{1}{\omega_c\tau} \end{pmatrix}$$

$$\text{or writing } \frac{\sigma_0}{\omega_c\tau} = \frac{ne^2\tau}{m^*} \frac{m^*c}{eH\tau} = \frac{ne c}{H} \quad \text{where}$$

$$\sigma_0 \text{ is Drude conductivity } \Rightarrow \vec{\sigma} = \sigma_0 \begin{pmatrix} (\frac{1}{\omega_c\tau})^2 & -\frac{1}{\omega_c\tau} \\ \frac{1}{\omega_c\tau} & (\frac{1}{\omega_c\tau})^2 \end{pmatrix}$$

(Compare with prob #1 on HW #1 !!)

$$\Rightarrow \text{resistivity tensor } \vec{\rho} = \vec{\sigma}^{-1} = \frac{1/\sigma_0}{(\frac{1}{\omega_c\tau})^4 + (\frac{1}{\omega_c\tau})^2} \begin{pmatrix} (\frac{1}{\omega_c\tau})^2 & +\frac{1}{\omega_c\tau} \\ -\frac{1}{\omega_c\tau} & (\frac{1}{\omega_c\tau})^2 \end{pmatrix}$$

~~All coefficient is $R = \frac{\rho_{xx}}{H}$~~ as $(\frac{1}{\omega_c\tau}) \ll 1$

$$\vec{\rho} = \frac{1/\sigma_0}{1 + (\frac{1}{\omega_c\tau})^2} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \approx \frac{1}{\sigma_0} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$$

$$\vec{j} = \vec{\sigma} \cdot \vec{E}$$

$$\vec{\sigma} = \frac{\sigma_0}{\omega_c \tau} \begin{pmatrix} 1 & -1 \\ \omega_c \tau & 1 \end{pmatrix}$$

$$\sigma_0 = \frac{ne^2\tau}{m^*}$$

$$\omega_c \tau = \frac{eH\tau}{m^*c} \gg 1$$

Then $\vec{E} = \vec{\rho} \cdot \vec{j}$ where $\vec{\rho} \approx \frac{1}{\sigma_0} \begin{pmatrix} 1 & \omega_c \tau \\ -\omega_c \tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}$

For $\vec{j} = j\hat{x}$ then $E_y = \rho_{yx}j = -\rho_{xy}j$

Hall coef: $R = \frac{E_y}{jH} = \frac{-\rho_{xy}}{H} = \frac{-\omega_c \tau}{\sigma_0 H} = \frac{-eH\tau}{m^*c} \frac{m^*}{ne^2\tau} \frac{1}{H}$
 $= -\frac{1}{mec}$ Drude value

~~For holes, $\vec{j} = m_h$~~

For electrons we used $\vec{j} = -me\vec{w} + \frac{me}{\omega_c \tau} \vec{H} \times \vec{w}$

For holes we use instead $\vec{j} = +m_h\vec{w} - \frac{m_h}{\omega_c \tau} \vec{H} \times \vec{w}$
 since charge carriers have charge $+e$.

All results carry through except take $e \rightarrow -e$

$$\Rightarrow R = \frac{1}{m_h ec}$$