Hall coefficient is
\[ R = -\frac{\sigma_{xy}}{\sigma_{xx}} \]  
(see Quantum Hall effect notes)

\[ = -\frac{\mu e T}{\sigma_0 H} = -\frac{eH}{m^*c} \frac{Te^*}{ne^2cH} = -\frac{1}{ne} \]  
as before.

Magnetic resistance
\[ \sigma_{xx} = \sigma_{xy} = \sigma_0 \]  
saturates to finite value as \( H \to 0 \)
just as was found in Drude model, 
extcept now \( m \) is new if there are 
several partially filled bands.

\[ \text{Case (2) } \]
Neither all occupied states, nor all unoccupied states 
have closed orbits \( \to \) in either electron or hole 
picture, there are open orbits we have to consider.

Now we will find that the \( \langle \hat{E} \rangle \) contribution to current 
\( \vec{j} \) from these open orbits no longer vanishes in the 
\( WcT \to \infty \) limit, and it dominates over the drift 
contribution to the current \( -ne \).

\[ \text{when } E=0, \hat{H} = H \hat{z} \text{ induces motion in orbits} \]
on the constant energy surfaces. An electron 
moving in an open orbit in \( k \)-space in the 
\( k_y \) direction, gives a current in real \( k \)-spacene 
the \( +x \) direction (rotate by 
90° about \( \hat{z} \)). However, when \( E=0 \), 
each occupied open orbit going in one 
direction is paired with an occupied 
open orbit going in the opposite 
direction, so the net current is 
zero.
Note: For an open orbit traveling along \( \hat{K}_y \), \( k_y(t) \) is periodic in time \( \Rightarrow V_y = \langle \frac{\partial E}{\partial k_y} \rangle = 0 \) averaged over time. But \( k_x(t) \) is constant + oscillation \( \Rightarrow V_x = \langle \frac{\partial E}{\partial k_x} \rangle \neq 0 \Rightarrow \) electron moves in \( \hat{x} \) direction.

Repeated zone scheme in \( k \)-space

when \( E \neq 0 \), in steady state, there will be an imbalance in occupation of open orbits, so that those orbits which absorb energy from the \( E \)-field have a larger population than those which lose energy to the field. (\( E \)-field heats up metal!)

\( E_x < 0 \Rightarrow \) net

\( V_x > 0 \Rightarrow J_x < 0 \)

so

\( J_x \sim E_x \) to lowest order in \( E \)

\( \vec{J} \sim \vec{x}(\vec{E} \cdot \vec{x}) \)

\( \langle \vec{J} \rangle \sim \vec{x}(\vec{E} \cdot \vec{x}) \)

Open orbits in \( \hat{y} \) direction have real space direction +\( \hat{x} \) \( \Rightarrow \) they gain energy from \( E \)-field if \( E_x < 0 \)

as energy absorbed is \(-e\vec{E} \cdot \vec{V} \) between collisions.

Open orbits in \( \hat{y} \) directions have real space direction -\( \hat{x} \) \( \Rightarrow \) they lose energy if \( E_x > 0 \).

\( \Rightarrow \) We assume therefore that the imbalance in occupation of open orbits in steady state gives rise to a current. If \( \vec{J} \) is the direction in real space of the open orbits, then the contribution to the current \( \vec{J} \) is in the \( \vec{J} \) direction, and proportional to some function of \( \vec{E} \cdot \vec{J} \).

\( \vec{J} \) open orbits \( \sim \vec{m} g(\vec{E} 
\cdot \vec{J}) \) \( \Rightarrow \) expand in small \( \vec{E} \).
Equivalently, since $\overline{E} = E - \hbar \mathbf{k} \cdot \mathbf{\hat{v}}$ is conserved between collisions, if $\Delta E = -e \overline{E} \cdot \mathbf{\hat{v}} \Delta t$ is energy absorbed by electron from $E$-field then

$$\Delta \overline{E} = 0 \Rightarrow \Delta E = \hbar \mathbf{\hat{v}} \cdot \Delta \mathbf{k}$$

So again we see in our example that it is the right hand open orbits moving along $+\mathbf{\hat{k}}$, that absorb energy, i.e $\mathbf{\hat{v}} \cdot \Delta \mathbf{k} > 0$ for these orbits, while $\mathbf{\hat{v}} \cdot \Delta \mathbf{k} < 0$ for left hand open orbits moving along $-\mathbf{\hat{k}}$.

So both $\mathbf{\hat{v}} \cdot \Delta \mathbf{k}$ and $-E \cdot \mathbf{\hat{v}}$ tell how much energy the electron absorbs from $E$-field.
This imbalance in steady state occupation of open orbits is determined by the quantity \(-e\vec{E} \cdot \vec{B}_C\), the energy absorbed by electrons from \(\vec{B}_C\) field in between collisions. If \(\hat{\mu}\) is real space direction of open orbit, \(\Rightarrow \langle \vec{v} \rangle \hat{\mu} \) in \(\hat{\mu}\) direction, so the current due to open orbits is in the \(\hat{\mu}\) direction, and is some function of \((\vec{E} \cdot \hat{\mu})\)

\[
\vec{j}_{\text{open}} = \hat{\mu} \cdot \vec{g}(\vec{E} \cdot \hat{\mu}) - \text{expanded for small } \vec{E}, \text{ using:}
\begin{align*}
\vec{j} &= 0 \text{ when } \vec{E} = 0, \text{ and} \\
\vec{j}(\vec{E}) &= -\vec{j}(-\vec{E})
\end{align*}
\]

\[
\vec{j}_{\text{open}} \sim \hat{\mu} (\vec{\hat{\mu}} \cdot \vec{E})
\]

where proportionality constant is independent of magnetic field \(H\).

We can write the contribution to conductivity tensor due to open orbits as

\[
\vec{j}_{\text{open}} = \hat{\sigma} \cdot \vec{E}
\]

where \(\hat{\sigma} = \lambda \sigma_0 \hat{\mu} \hat{\mu}
\]
constant indep of \(H\)

If we choose \(\hat{\mu}\) in \(x\) direction,

\(\hat{\sigma} = \lambda \sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\)

If we treat the contribution to conductivity tensor from closed orbits as before, we get for total conductivity tensor

\[
\hat{\sigma} = \frac{\sigma_0}{(\omega_C)^2} \begin{pmatrix} 1 - \omega_C^2 & 0 \\ \omega_C^2 & 1 \end{pmatrix} + \lambda \sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \sigma_0 \begin{pmatrix} \lambda + (\omega_C)^2 & -\frac{1}{\omega_C^2} \\ \frac{1}{\omega_C^2} & \frac{1}{(\omega_C^2)^2} \end{pmatrix}
\]
\( \sigma \) resistivity tensor 

\[
\sigma^{-1} = \frac{1}{\sigma_0} \left( \frac{1}{(w_c t)^2 + \frac{1}{w_c t} + \frac{1}{w_c t} + \frac{1}{w_c t}^4} \right)
\]

\[
\approx \frac{1}{\sigma_0 (1+\lambda)} \left( \begin{array}{cc}
\frac{1}{w_c t} & \frac{w_c t}{\lambda (w_c t)^2 + 1} \\
-w_c t & \frac{w_c t}{\lambda (w_c t)^2 + 1}
\end{array} \right)
\]

Note \( S_{xy} = -S_{yx} \) as before for closed orbits, and Hall coefficient is \( \frac{S_{xy}}{H} \sigma_0 (1+\lambda) = \frac{-1}{\text{nec} (1+\lambda)} \) same as before except for factor \( (1+\lambda) \).

But now \( S_{xx} \neq S_{yy} \). We have

\[ S_{xx} \quad \text{magneto-resistance for current flowing \( II \) to open orbits in real space (i.e. \( \vec{j} = \vec{j}_x \))} \]

\[
\approx \frac{1}{\sigma_0 (1+\lambda)} \quad \text{saturates as} \ H \to \infty \ \text{as in Drude mode}
\]

\[ S_{yy} \quad \text{magneto-resistance when current flowing \( \vec{j} \) to direction of open orbits in real space (i.e. \( \vec{j} = \vec{j}_y \))} \]

\[
= \frac{\lambda}{\sigma_0 (1+\lambda)} (w_c t)^2 \sim H^2 \quad \text{does not saturate as} \ H \to \infty, \ \text{grows as} \ H^2
\]

magneto-resistance which keeps increasing with \( H \) is signal for presence of open orbits on Fermi surface.
For a current in a general direction \( \mathbf{j} = j(\cos \theta, \sin \theta) \), where \( \theta \) measures angle from \( \mathbf{x} \), the direction of the open orbits in real space, we have

\[
\mathbf{E} = \mathbf{k} \cdot \mathbf{j} = j \frac{\mathbf{k}}{\sigma_0(1+\lambda)} \begin{pmatrix} \cos \theta + (\omega c j) \sin \theta \\ - (\omega c j) \cos \theta + (\lambda (\omega c j)^2 + 1) \sin \theta \end{pmatrix}
\]

and the longitudinal magnetoresistance is

\[
\sigma = \frac{\mathbf{E} \cdot \mathbf{E}}{|\mathbf{E}|^2}
\]

\[
= \frac{1}{\sigma_0(1+\lambda)} \left[ \cos^2 \theta + (\omega c j) \sin \theta \cos \theta \\ - (\omega c j) \cos \theta \sin \theta + \lambda (\omega c j)^2 + 1 \right] \sin \theta
\]

\[
\sigma = \frac{1}{\sigma_0(1+\lambda)} \left[ 1 + \lambda (\omega c j)^2 \sin^2 \theta \right]
\]

- Constant.
- Inside the point from closed orbits
- Increases without bound as \( J \) increases from open orbits.
Lattice vibrations, phonons, and the speed of sound.

Assume Hamiltonian of conic degrees of freedom looks like

\[ H = \sum \frac{\vec{p}_i^2}{2M} + U_{\text{ion}}(\frac{\vec{R}_i}{\vec{R}_i^0}) \]

lattice potential due to ion-ion interactions

ions at positions \( \vec{R}_i \), momentum \( \vec{p}_i \), mass \( M \)

Write \( \vec{R}_i = \vec{R}_i^0 + \vec{u}_i \)

\( \uparrow \)

position in periodic BL small displacement due to elastic distortions

If \( \vec{u}_i \) is small, expand \( U_{\text{ion}} \) about the BL positions \( \vec{R}_i^0 \). Since the positions \( \vec{R}_i^0 \) are assumed to be positions of mechanical equilibrium, the linear term in the expansion must vanish, and the quadratic term is the leading order term.

\[ U_{\text{ion}}(\frac{\vec{R}_i}{\vec{R}_i^0}) = U_{\text{ion}}^0 + \frac{1}{2} \sum \frac{\partial^2 U_{\text{ion}}}{\partial \vec{u}_i \partial \vec{u}_j} \vec{u}_i \cdot \vec{u}_j \]

\( i, j \) label BL sites
\( x, y, z \) label components \( x, y, z \) of the displacement

\[ D_{ij}^{x\beta} = \frac{\partial^2 U_{\text{ion}}}{\partial \vec{u}_i^x \partial \vec{u}_j^\beta} \]

\( x, \beta \) the dynamical matrix
The classical equations of motion for the ions are then

\[ M \dddot{\mathbf{U}}_i = -\nabla \Omega \mathbf{U}_i \implies M \dddot{\mathbf{U}}_{id} = -\sum D^{\alpha \beta}_{ij} \mathbf{U}_{id} \]

Now by translational invariance of the Bravais lattice, \( D^{\alpha \beta}_{ij} \) depends only on \( \mathbf{R}_i - \mathbf{R}_j \).

We can define the Fourier transforms

\[ \tilde{\mathbf{U}}_i(t) = \int d^3q \int dw e^{i \mathbf{q} \cdot \mathbf{R}_i} e^{-iwt} \mathbf{U}(\mathbf{q}, w) \]

\[ \tilde{\mathbf{U}}_i \in 1^{st} BZ \]

\[ D^{\alpha \beta}_{ij} = \int d^3q e^{i \mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} D^{\alpha \beta}(\mathbf{q}) \]

\[ D^{\alpha \beta}_{ij} \in 1^{st} BZ \]

---

Note: in defining Fourier transform of a function that exists only on the discrete sites of a B.Z., the only wave vectors we need to consider are those \( \mathbf{q} \) in the 1st B.Z. This is because any wave vector \( \mathbf{q} \) can always be written as \( \mathbf{q} = \mathbf{Q} + \mathbf{K} \) with \( \mathbf{K} \) a unique B.Z. vector and \( \mathbf{Q} \) in the 1st B.Z. Then the plane wave factor would be

\[ e^{i \mathbf{K} \cdot \mathbf{R}_i} = e^{i (\mathbf{Q} + \mathbf{K}) \cdot \mathbf{R}_i} = e^{i \mathbf{Q} \cdot \mathbf{R}_i} \]  \( \text{since} \) \( e^{i \mathbf{K} \cdot \mathbf{R}_i} = 1 \)

so we still only get oscillations at \( \mathbf{Q} \) in 1st B.Z.
Substitute these into the equation of motion

\[ \int d^3q \int_0^{\infty} d\omega e^{\hat{q} \cdot \vec{R}_i^0 - \omega t} (-\omega^2) M \overrightarrow{U}(q, \omega) \]

\[ = \int d^3q \int d^3q' \sum_{g \in 1^{st} BZ} e^{i \hat{q} \cdot (\vec{R}^0 - \vec{R}_j^0)} e^{i \hat{q}' \cdot \vec{R}_j^0} e^{-i\omega t} \]

Do the integral summation over coordinates

\[ \sum_{j} e^{i (\hat{\theta}' - \hat{\theta}) \cdot \vec{R}_j^0} = \delta (\hat{\theta}' - \hat{\theta}) \]

Follows since \( \{ \vec{R}_j^0 + \vec{R}_j^0 \} \) = \( \vec{R}^0 \) \( \delta \) since B.L. is closed under translation by any B.L. vector \( \vec{R}_0^0 \)

\[ \Rightarrow \sum_{j} e^{i (\hat{\theta}' - \hat{\theta}) \cdot \vec{R}_j^0} = \sum_{\vec{R}_j^0} e^{i (\hat{\theta}' - \hat{\theta}) \cdot (\vec{R}_j^0 + \vec{R}_0^0)} \]

\[ = e^{i (\hat{\theta}' - \hat{\theta}) \cdot \vec{R}_0^0} \sum_{\vec{R}_j^0} e^{i (\hat{\theta}' - \hat{\theta}) \cdot \vec{R}_j^0} \]

\[ \Rightarrow e^{i (\hat{\theta}' - \hat{\theta}) \cdot \vec{R}_0^0} = 1 \text{ for any } \vec{R}_0^0 \text{ in B.L.} \]

\[ \Rightarrow \hat{\theta}' - \hat{\theta} = \vec{K} \text{ in } R, L. \]

But since \( \hat{\theta}, \hat{\theta}' \) both in 1st BZ \( \Rightarrow \vec{K} = 0 \)

and \( \hat{\theta} = \hat{\theta}' \) or the sum must vanish
\[
\int d^3q \int dw \, e^{i(\vec{q} \cdot \vec{r}_i - \omega t)} \left( -\omega^2 M \hat{u}(\vec{q}, \omega) \right) \\
= -\int d^3q \int dw \, e^{i(\vec{q} \cdot \vec{r}_i - \omega t)} \left( \frac{\vec{D}(\vec{q})}{\omega} \cdot \vec{u}(\vec{q}, \omega) \right)
\]

Squash Fourier amplitudes to get

\[\omega^2 M \hat{u}(\vec{q}, \omega) = \vec{D}(\vec{q}) \cdot \vec{u}(\vec{q}, \omega)\]

If the eigenvectors and eigenvalues of \(\vec{D}(\vec{q})\) are \(\vec{E}_1(\vec{q}), \vec{E}_2(\vec{q}), \vec{E}_3(\vec{q})\) and \(\lambda_1(\vec{q}), \lambda_2(\vec{q}), \lambda_3(\vec{q})\)

Then

\[\omega_s^2 M = \lambda_s(\vec{q}) \quad s = 1, 2, 3\]

\[\omega_s = \sqrt{\frac{\lambda_s(\vec{q})}{M}}\]

dispersion relation for elastic vibrations at wave vector \(\vec{q}\), polarization \(\vec{E}_s(\vec{q})\)

We expect that in the long wave length limit we can expand

\[\vec{D}(\vec{q}) = \sum_i e^{-i \vec{q} \cdot \vec{R}_i} \vec{D}(\vec{R}_i)\]

\[\approx \sum_i \left( 1 - i \vec{q} \cdot \vec{R}_i \right) \vec{D}(\vec{R}_i)\]
\[ \sum_i \mathbf{D}(\mathbf{R}_i) = 0 \] because at all \( \mathbf{U}_i = \mathbf{U}_0 \) a uniform displacement, hence net force on \( \mathbf{U}_0 \) must vanish.

\[ \sum_i \mathbf{R}_i \cdot \mathbf{D}(\mathbf{R}_i) = 0 \] by inversion symmetry \( \mathbf{R}_i \rightarrow -\mathbf{R}_i \)

\[ \mathbf{D}(\mathbf{R}) = - \frac{1}{2} \sum \mathbf{R}_i \cdot (\mathbf{q} \cdot \mathbf{R}_i) \mathbf{D}(\mathbf{R}) \]

we assume this sum converges

\[ \Rightarrow \mathbf{D}(\mathbf{q}) \propto \mathbf{q}^2 \]

so \( \lambda_s(\mathbf{q}) \propto \mathbf{q}^2 \) or \( \lambda_s(\mathbf{q}) = \frac{A_s}{M} \mathbf{q}^2 \) for small \( \mathbf{q} \)

\[ \Rightarrow w_s = \sqrt{\frac{A_s}{M}} |\mathbf{q}| \] with

\[ c_s = \sqrt{\frac{A_s}{M}} \] the speed of sound for polarization s.

\[ w_s = c_s q \] for small \( \mathbf{q} \)

Also at small \( \mathbf{q} \) we expect the spatial orientation of the B.L. to get "averaged over" and so the only directions of \( \mathbf{q} \) and \( \mathbf{q} \rightarrow \mathbf{q} \). We thus expect the polarization vectors to become as \( \mathbf{q} \rightarrow 0 \)

\[ \mathbf{E}_1(\mathbf{q}) = \mathbf{q} \] longitudinal sound mode, speed \( c_1 \)

\[ \mathbf{E}_2(\mathbf{q}) \parallel \mathbf{q} \] transverse sound modes, speed \( c_2 \).