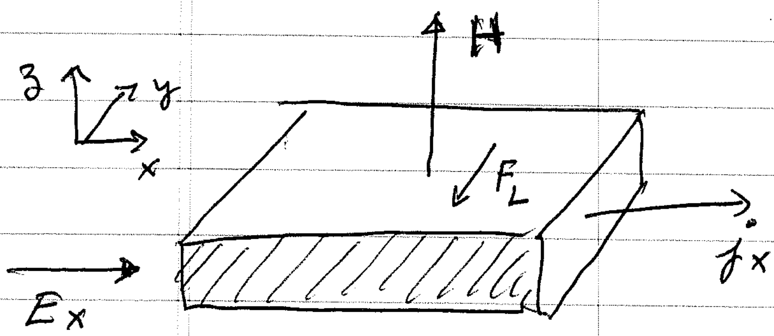
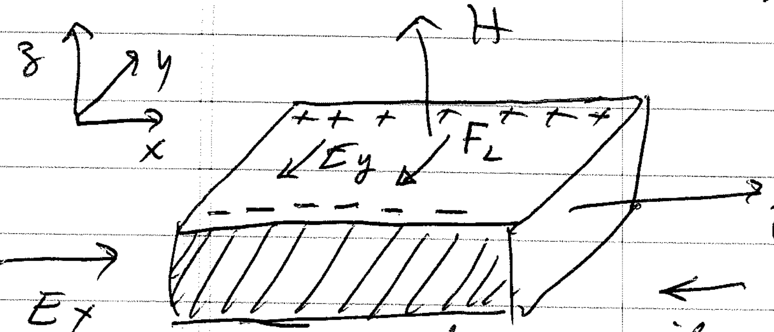
Hall effect (1879) – determines the sign of the charges that carry the electric current in a metal

electron motion in combined static electric and magnetic fields



Electric field E_x applied in $\hat{x}$ direction produces flowing electric current j_x in $\hat{x}$ direction. Magnetic field H in $\hat{z}$ direction exerts Lorentz force $\frac{\vec{j}}{c}\times\vec{H}$ on the ~~charges car~~ moving charges carrying the current $\vec{j}$. For $\vec{j}$ in $\hat{x}$ direction and $\hat{H}$ in $\hat{z}$ direction, the Lorentz force $\vec{F}_L$ is in the $-\hat{y}$ direction. $\vec{F}_L$ deflects the charge carriers to the side wall of the wire (the shaded wall in the figure) where they build up and create a surface charge density. The surface charge density produces an electric field E_y in $\hat{y}$ direction. For a steady state situation, the force from E_y will exactly cancel out the Lorentz force F_L. If W is the width of the wire, then measuring the "Hall voltage" $V_y = E_y W$ allows one to determine the sign of the charges that carry the electric current.

If current is carried by negative charges $-q$, then

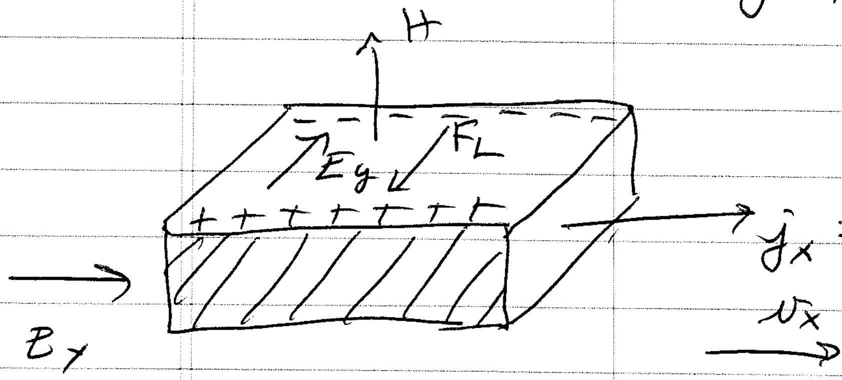


$j_x = -q n v_x \Rightarrow v_x < 0$

F_L deflects the mobile negative charges carrying the current, and negative charges build up on shaded surface
(neutrality of system $\Rightarrow$ absence of negative charge, ie positive charge, builds up on opposite surface)

The electric field E_y is in $-\hat{y}$ direction
and Hall voltage is <u>negative</u>

If current is carried by positive charges $+q$, then



$j_x = q n v_x$, $v_x > 0$

F_L deflects the mobile positive charges carrying the current and positive charge builds up on the shaded surface

The electric field E_y is in the $+\hat{y}$ direction
and the Hall voltage is <u>positive</u>

For most (but not all) metals one finds a negative Hall voltage. This established that it was negatively charged electrons that carry the electric current in most metals.

<u>Quantities to measure</u>

<u>Hall coefficient</u> $R_H \equiv \dfrac{E_y}{j_x H}$

since we expect force from E_y to exactly balance out Lorentz force F_L in steady state, we expect R_H to be independent of H

<u>magnetoresistance</u> $\rho(H) \equiv \dfrac{E_x}{j_x}$

We can compute both R_H and ρ using the Drude model.

$$\frac{d\vec{p}}{dt} = -e\left(\vec{E} + \frac{\vec{p}}{mc} \times \vec{H}\right) - \frac{\vec{p}}{\tau} = 0 \text{ in steady state}$$

for x and y components

$$0 = -eE_x - \frac{eH}{mc} p_y - \frac{p_x}{\tau}$$

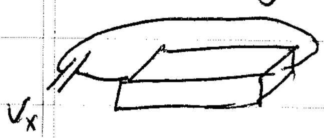
$$0 = -eE_y + \frac{eH}{mc} p_x - \frac{p_y}{\tau}$$

$\omega_c \equiv \dfrac{eH}{mc}$ cyclotron frequency = angular frequency of a charged particle in circular motion in uniform H

① $eE_x = -\omega_c p_y - \frac{p_x}{\tau}$

② $eE_y = \omega_c p_x - \frac{p_y}{\tau}$

In steady state, current flows only in $\hat{x}$ direction.



No current flows out the side walls of the wire $\Rightarrow p_y = 0$

with $p_y = 0$,

① $\Rightarrow p_x = -eE_x\tau$

$$j_x = -neV_x = -\frac{nep_x}{m} = \frac{ne^2\tau}{m}E_x$$

$$\boxed{\frac{E_x}{j_x} = \frac{m}{ne^2\tau} = \rho}$$

magnetoresistance $\rho(H) = \frac{1}{\sigma} = \frac{m}{ne^2\tau}$

same as ordinary d.c. resistivity ρ when $H=0$

In Drude model, $\rho(H)$ is <u>independent</u> of H!
agreed with expt measurements by Drude.
More modern expts however do find ρ can vary with H.

② $\Rightarrow E_y = \frac{\omega_c}{e} p_x = -\omega_c \tau E_x$

Hall coefficient $R_H = \frac{E_y}{j_x H} = \frac{\left(\frac{\omega_c}{e} p_x\right)}{\left(-\frac{nep_x}{m}\right) H} = \frac{-\omega_c m}{ne^2 H}$

use $\omega_c = \frac{eH}{mc} \Rightarrow R_H = \frac{-\left(\frac{eH}{mc}\right) m}{ne^2 H} = -\frac{1}{nec}$

$$\boxed{R_H = -\frac{1}{nec}}$$ Hall coefficient independent of H

But also, R_H is independent of our phenomenological parameter τ, the relaxation time.

R_H is something we can directly test against experiment since it only depends on the electron density n, which can be easily calculated.

In practice R_H is found to depend on H and T and also on sample preparation. But at low T, high H ($\sim 10^4$ gauss) very pure samples, R_H is found to approach a constant value, often very close to the Drude value

	metal	valence	$-1/R_H nec$ (=1 for Drude)	
alkali metals	Li	1	0.8	Drude prediction very good for alkali metals which have single S shell electron as valence electron
	Na	1	1.2	
	K	1	1.1	
	Rb	1	1.0	
	Cs	1	0.9	
noble metals filled d-levels	Cu	1	1.5	
	Ag	1	1.3	
	Au	1	1.5	
	Be	2	-0.2	sign is negative! $\Rightarrow$ current is carried by objects with <u>positive</u> sign
	Mg	2	-0.4	
	In	3	-0.3	
	Al	3	-0.3	

ac electric conductivity

$$\vec{E}(t) = Re\left[\vec{E}_\omega e^{-i\omega t}\right]$$ harmonic oscillating electric field

$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e\vec{E}(t)$$ Drude eqn of motion

assume solution is also harmonic oscillation

$$\vec{p}(t) = Re\left[\vec{p}_\omega e^{-i\omega t}\right]$$

$$\Rightarrow \quad -i\omega\vec{p}_\omega = -\frac{\vec{p}_\omega}{\tau} - e\vec{E}_\omega$$

$$\left(\frac{1}{\tau} - i\omega\right)\vec{p}_\omega = -e\vec{E}_\omega$$

$$\vec{p}_\omega = \frac{-e}{\frac{1}{\tau} - i\omega}\vec{E}_\omega = \frac{-e\tau}{1-i\omega\tau}\vec{E}_\omega$$

current $\vec{j}(t) = Re\left[\vec{j}_\omega e^{-i\omega t}\right]$ $\quad \vec{j} = -en\vec{v} = -en\frac{\vec{p}}{m}$

$$\vec{j}_\omega = -en\frac{\vec{p}_\omega}{m}$$

$$= \frac{ne^2\tau}{m(1-i\omega\tau)}\vec{E}_\omega$$

a.c. conductivity

$$\vec{j}_\omega = \sigma(\omega)\vec{E}_\omega$$

$$\Rightarrow \quad \sigma(\omega) = \frac{ne^2\tau}{m(1-i\omega\tau)} = \frac{\sigma_{dc}}{1-i\omega\tau}$$

where $\sigma_{dc} = \frac{ne^2\tau}{m}$ is d.c. Drude conductivity

as $\omega \to 0$, $\sigma(\omega) \to \sigma_{dc}$

as $\omega \to \infty$, $\sigma(\omega) \to \frac{ne^2\tau}{-i\omega\tau m} = \frac{ine^2}{m\omega}$ indep of τ

for $\omega\tau \gg 1$, ie $\omega \gg \frac{1}{\tau}$, oscillation is fast compared to collision rate, so $\sigma(\omega)$ becomes independent of τ.

Electromagnetic wave propagation in a metal

approx 1) In CGS units, for a propagating electromagnetic plane wave $|\vec{E}| = |\vec{H}|$.
So for the forces the EM wave exerts on a conduction electron

$$\frac{|\vec{F}_{mag}|}{|\vec{F}_{elec}|} = \frac{-e\,|\frac{\vec{v}}{c} \times \vec{H}|}{-e|\vec{E}|} \sim \frac{v}{c} \ll 1$$

so we will ignore the force that the $\vec{H}$ component of the wave exerts on the electron

approx 2) When wavelength λ of EM wave is much larger than mean free path l of collisions, $\lambda \gg l$, the electric field that an electron sees over the time between collisions is roughly ~~consta~~ uniform in space. Good for waves in visible spectrum where $\lambda \sim 5000\,\text{Å}$, $l \sim 10\,\text{Å}$.

(1) + (2) $\Rightarrow$ we can use the above computed a.c. conductivity $\sigma(\omega)$ to find the relation

between the electric field of the EM wave and the resulting current due to the conduction electrons

For a single harmonic electromagnetic wave we can write for the fields:

$$\vec{E}(\vec{r},t) = Re\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] \quad \text{electric field}$$
$$\vec{H}(\vec{r},t) = Re\left[\vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] \quad \text{magnetic field}$$

The relation between the amplitudes $\vec{E}_0$ and $\vec{H}_0$ and between the wavevector $\vec{k}$ and frequency ω are determined by Maxwell's Equations.

We will look for solutions for a transverse propagating wave, ie with $\vec{E}_0 \perp \vec{k}$.

Macroscopic Maxwell's Equations (in CGS units)

Gauss $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$ Gauss

Ampere $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ Faraday

We will ignore magnetization effects, ie $\mu = 1$ and $\vec{B} = \vec{H}$

We will ignore polarization from bound electrons, ie $\varepsilon = 1$ and $\vec{E} = \vec{D}$

$\vec{j}$ is current due to conduction electrons

ρ is any locally non-neutral charge density due to variations in conduction electron density

Above become

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}$$

Substitute into the above the single harmonic forms for $\vec{E}$ and $\vec{H}$.

Gauss $\quad \vec{\nabla}\cdot\vec{E} = i\vec{k}\cdot\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0 \quad$ since assumed $\vec{E}_0 \perp \vec{k}$

$\Rightarrow \rho = 0 \quad$ transverse EM wave induces no local charge density

Gauss $\quad \vec{\nabla}\cdot\vec{H} = i\vec{k}\cdot\vec{H}_0 e^{i(k\cdot r-\omega t)} = 0$

$\Rightarrow \vec{H}_0 \perp \vec{k} \quad$ so magnetic field is also transverse

(1) Faraday $\quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \Rightarrow i\vec{k}\times\vec{E}_0 = i\frac{\omega}{c}\vec{H}_0$

(2) Ampere $\quad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \Rightarrow i\vec{k}\times\vec{H}_0 = \frac{4\pi}{c}\vec{j}_0 - \frac{i\omega}{c}\vec{E}_0$

where assumed $\vec{j}(\vec{r},t) = \mathrm{Re}\left[\vec{j}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$

Multiply (1) by $\vec{k}\times$

$$i\vec{k}\times(\vec{k}\times\vec{E}_0) = \underbrace{i(\vec{k}\cdot\vec{E}_0)\vec{k}}_{=0 \text{ since } \vec{k}\perp\vec{E}_0} - ik^2\vec{E}_0 = \frac{i\omega}{c}\vec{k}\times\vec{H}_0$$

so $\quad i\vec{k}\times\vec{H}_0 = -i\frac{k^2c}{\omega}\vec{E}_0$

Substitute from (2)

$$i\vec{k}\times\vec{H}_0 = -i\frac{k^2c}{\omega}\vec{E}_0 = \frac{4\pi}{c}\vec{j}_0 - \frac{i\omega}{c}\vec{E}_0$$

$$\Rightarrow \quad \vec{E}_0\left(k^2 - \frac{\omega^2}{c^2}\right) = \frac{4\pi i\omega}{c^2}\vec{j}_0$$

For a vacuum, $\vec{j}_0 = 0 \quad \Rightarrow \quad \omega^2 = c^2 k^2$

For a metal $\quad \vec{j}_0 = \sigma(\omega)\vec{E}_0 \qquad \sigma(\omega)$ is ac conductivity

$$\Rightarrow \quad \vec{E}_0\left(k^2 - \frac{\omega^2}{c^2}\right) = \frac{4\pi i\,\omega}{c^2}\sigma(\omega)\vec{E}_0$$

$$\Rightarrow \quad \vec{E}_0\left(k^2 - \frac{\omega^2}{c^2} - \frac{4\pi i\omega}{c^2}\sigma(\omega)\right) = 0$$

$$\Rightarrow \quad \boxed{k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi i\,\sigma(\omega)}{\omega}\right)}$$

← "dispersion relation" for EM waves in a metal.

$$\sigma(\omega) = \frac{\sigma_{dc}}{1 - i\omega\tau} \qquad \sigma_{dc} = \frac{ne^2\tau}{m}$$

For low frequencies, $\omega\tau \ll 1$, ie frequency much smaller than collision rate, $\sigma(\omega) \approx \sigma_{dc}$

then $\quad k^2 = \dfrac{\omega^2}{c^2}\left(1 + \dfrac{4\pi i\,\sigma_{dc}}{\omega}\right)$

For fixed real ω, k will have a large imaginary part. When ω low enough that $4\pi\sigma_{dc}/\omega \gg 1$ then

$$k^2 = \frac{\omega^2}{c^2}\frac{4\pi i\,\sigma_{dc}}{\omega}$$

$$k = \frac{1}{c}\sqrt{4\pi\sigma_{dc}\,\omega}\left(\frac{1+i}{\sqrt{2}}\right)$$

k is complex number with equal real and imaginary parts.

since $\vec{E} = Re\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$

if we write $\vec{k} = \vec{k}_1 + i\vec{k}_2$

$$E = Re\left\{\vec{E}_0 e^{i(\vec{k}_1\cdot\vec{r}-\omega t)}\right] e^{-\vec{k}_2\cdot\vec{r}}$$

wave decays as it propagates into metal

In low freq limit where $k_1 \approx k_2$, wave ~~propo~~ decays by factor 1/e for every wavelength it penetrates.

More interesting behavior in higher frequency limit, $\omega\tau \gg 1$, ie frequency large compared to collision rate.

then $\sigma(\omega) \approx \dfrac{\sigma_{dc}}{-i\omega\tau} = \dfrac{ne^2\tau}{m\omega\tau}\,i = \dfrac{ne^2}{m\omega}\,i$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{4\pi ne^2}{m\omega^2}\right)$$

call $\omega_p \equiv \sqrt{\frac{4\pi n e^2}{m}}$ the "plasma frequency"

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)$$ dispersion relation is <u>independent</u> of τ

In general $\omega_p \gg \frac{1}{\tau}$

For freq $\omega > \omega_p$ we have

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)$$ is positive real number

$$k = \frac{\omega}{c}\sqrt{1 - (\omega_p/\omega)^2}$$ is real

EM wave propagates through the metal with no attenuation. For $\omega > \omega_p$, the metal is <u>transparent</u> to EM waves!

But for freq $\frac{1}{\tau} \ll \omega < \omega_p$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right) < 0$$ is negative

$$k = i\frac{\omega}{c}\sqrt{\left|1 - (\omega_p/\omega)^2\right|}$$ is pure imaginary

$$\vec{E}(\vec{r},t) = \cancel{\mathrm{Re}\{E_0 e^{ikr}}$$

$$\mathrm{Re}\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]$$

$$= \mathrm{Re}\left[\vec{E}_0 e^{-i\omega t}\right] e^{-|k|\hat{n}\cdot\vec{r}}$$

field decays exponentially — waves do not propagate

$\omega < \omega_p$, EM waves get absorbed by the metal

The crossover from absorption to transparent occurs at $\omega = \omega_p$, or at wavelength

$$\lambda_p \equiv 2\pi c/\omega_p \qquad \omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

depends only on electron density.

Compare this Drude prediction to experiment

metal	λ_p (Drude) (10^3 Å)	λ_p (expt) (10^3 Å)
Li	1.5	2.0
Na	2.0	2.1
K	2.8	3.1
Rb	3.1	3.6
Cs	3.5	4.4

agreement is not bad given all the simplifying approximations that we have made!