In the preceeding calculations, we treated the paramagnetic and diamagnetic effects separately. ie, when computing Pauli paramagnetism we ignore the change in electron wave function due to the presence of the magnetic field H, and only considered the interaction of H with the intrinsic electron magnetic moment μ_{os}. When computing Landau diamagnetism we ignored this interaction with the intrinsic moment, and considered only the effect of H on the eigenstates and hence the density of states

Of course both effects are there simultaneously, so the total magnetic susceptibility of the free electron gas is therefore

$$\chi = \chi_p + \chi_L = \chi_p - \frac{1}{3}\chi_L = \frac{2}{3}\chi_p$$

Since $\chi_p > 0$, the net effect is <u>paramagnetic</u>

For some more traditional calculations of Landau diamagnetism see:

Notes from AP Young UC Santa Cruz
  http://bartok.ucsc.edu/peter/231/magnetic_field/node5.html
Pathria - "Statistical Mechanics", pgs 206 - 209
Landau + Lifshitz - "Statistical Mechanics V1", pgs 172 - 175

In real metals and doped semiconductors one does not necessarily find that the total χ is paramagnetic. When we include effects of band structure, which causes the electron to behave as if it had an "effective" mass m^*, rather than its true free particle mass m, we can find that χ can be either paramagnetic or diamagnetic

The Pauli paramagnetic susceptibility was

$$\chi_p \sim g(\varepsilon_F) = \frac{m k_F}{\hbar^2 \pi^2} \quad \text{for free electrons}$$

For electrons in a periodic potential, χ_p will go as $\sim m^*$ rather than m, so

$$\chi_p \sim \frac{m^*}{m} \chi_p^{\text{free electron}}$$

The Landau diamagnetic susceptibility came from the Lorentz force on the electron in a magnetic field
$$\chi_L \sim \frac{e \vec{v}}{c} \times \vec{H}$$

Where velocity $v \sim \frac{d\varepsilon_k}{dk} \sim \frac{1}{m}$ For electrons in a periodic potential we will thus have $v \sim \frac{1}{m^*}$ so
$$\chi_L \sim \frac{m}{m^*} \chi_L^{\text{free electron}}$$

So $\frac{\chi_L}{\chi_p} \sim \left(\frac{m}{m^*}\right)^2 \frac{\chi_L^{\text{free}}}{\chi_p^{\text{free}}}$ Since we can have

$m^* \ll m$ in some materials, it is possible to have $|\chi_L| \gg \chi_p$ and the material will have a net diamagnetic response

The de Haas – van Alphen effect

At sufficiently low temperature and high magnetic field, so that $\hbar w_c > k_B T$, the oscillations due to the discrete Landau levels can be observed in measurements of magnetization $M = -\frac{1}{V} \partial E / \partial H$. These were first observed by de Haas and van Alphen in 1930 in magnetization measurements on Bi at $14.2\,°K$. Similar oscillations are found in susceptibility $\chi = \partial M / \partial H$, conductivity (Shubnikov – de Haas effect), and many other quantities. Since we found that ε_F has such oscillations, so $g(\varepsilon_F)$ will have such oscillations, hence we can easily see why many physical quantities also oscillate.

The period of oscillations is in the <u>inverse</u> magnetic field $1/H$

$$\text{period is } \Delta X^\circ = 1 \quad \Rightarrow \quad \Delta\left(\frac{\varepsilon_F^\circ}{\hbar w_c}\right) = 1 \qquad w_c = \frac{eH}{mc}$$

since ε_F° is fixed while H varies, we have oscillations that are periodic in $1/H$ with period

$$\Delta\left(\frac{1}{H}\right) = \frac{\hbar}{\varepsilon_F^\circ}\frac{e}{mc}$$

we can rewrite this as

$$\Delta\left(\frac{1}{H}\right) = \frac{\hbar^2 m}{\hbar^2 k_F^2} \frac{e}{mc} = \frac{2e}{\hbar c k_F^2}$$

cross sectional area of the Fermi sphere
is $A_F = \pi k_F^2$, so

$$\boxed{\Delta\left(\frac{1}{H}\right) = \frac{2\pi e}{\hbar c} \frac{1}{A_F}}$$

The above turns out to be more generally true.
For electrons in a periodic potential (as opposed
to our free electron model) the Fermi surface
is not necessarily a sphere. Still the above
relation holds where A_F is the ~~maximal~~ extremal
cross sectional area of the Fermi surface
perpendicular to the direction of the applied
magnetic field. The de Haas - van Alphen
effect thus became one of the methods
for measuring the shape of the Fermi surface.

see Ashcroft & Mermin Chpt 14 for more details

<u>Screening</u> and the <u>dielectric function</u> of the
free electron gas

If an external electrostatic potential $V(\vec{r})$
is applied to the electron gas, the average
electron density will no longer be constant;
as electrons will get attracted to the maxmin
of V (so that $-eV$ is minimized).

If we write the new spatially varying ~~charge~~
electron density as

$$n(\vec{r}) = n + \delta n(\vec{r})$$

where n is the uniform density when $V = 0$, then

$$\delta \rho = -e \, \delta n(\vec{r}) \quad \text{is called the "induced"}$$

charge density.

The induced charge density $\delta \rho$ creates an
"induced" electrostatic potential of its own, δV,
given by,

$$-\nabla^2 \delta V = 4\pi \, \delta \rho \qquad \text{Poisson's eqn for electrostatics}$$

So the total potential is then

$$V^{tot} = V + \delta V$$

Consider this in Fourier transform space

$$V(\vec{k}), \quad V^{tot}(\vec{k}), \quad \delta V(\vec{k}), \quad \delta\rho(\vec{k}) \text{ are the}$$

Fourier transforms of $V(\vec{r})$, $V^{tot}(\vec{r})$, $\delta V(\vec{r})$, $\delta\rho(\vec{r})$

$$\text{ie} \quad V(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} V(\vec{k}) \quad \Big\} \quad \text{etc.}$$

$$V(\vec{k}) = \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}} V(\vec{r}) \quad \Big\}$$

The dielectric function $\varepsilon(\vec{k})$ is then defined by

$$V^{tot}(\vec{k}) = \frac{V(\vec{k})}{\varepsilon(\vec{k})}$$

[In your EM course you probably saw the dielectric
constant ε, defined by $D = \varepsilon E$. Here we
generalize this to spatially varying situations
by defining $\varepsilon(\vec{k})$. The limit $\varepsilon(k \to 0)$ is the
uniform dielectric constant you saw in EM
recall D is the field produced by the "free"
charge, ie the charge external to the material
itself, while E includes the fields due to the
polarized charges that make up the material.
Hence $\vec{E} = -\vec{\nabla} V^{tot}$, while $\vec{D} = -\vec{\nabla} V$ the
applied potential]

Now $\quad -\nabla^2 \delta V = 4\pi \delta \rho$

$$\Rightarrow \quad k^2 \delta V(\vec{k}) = 4\pi \, \delta\rho(\vec{k})$$

$$\delta V(\vec{k}) = \frac{4\pi}{k^2} \delta\rho(\vec{k})$$

$$V^{tot}(\vec{k}) = V(\vec{k}) + \delta V(\vec{k}) = V(\vec{k}) + \frac{4\pi}{k^2} \delta\rho(\vec{k})$$

So $\quad V(\vec{k}) = V^{tot}(k) - \frac{4\pi}{k^2} \delta\rho(\vec{k})$

$$\frac{V(\vec{k})}{V^{tot}(\vec{k})} = \varepsilon(\vec{k}) = 1 - \frac{4\pi}{k^2} \frac{\delta\rho(\vec{k})}{V^{tot}(\vec{k})}$$

We now need to find what is the $\delta\rho$ that is induced in the presence of a V^{tot}.
(Note: the electrons see not just the applied potential V, but also the induced potential δV. Hence, when considering the behavior of an isolated electron, it sees the total potential $V^{tot} = V + \delta V$, where δV describes the interaction of the electron under consideration with all the other electrons. This is an example of a mean field approximation — we will approximate the interaction of an electron will all other electrons via the average electrostatic potential δV created by the other electrons.)

To compute $\delta\rho$ we assume $V(\vec{r})$ is slowly varying so that the system is in local equilibrium at every position $\vec{r}$. This approx is good for getting the small $\vec{k}$ limit of $\varepsilon(\vec{k})$.

Then the probability to have an electron with wave vector $\vec{k}$ at position $\vec{r}$ is given by the Fermi function

$$f(\vec{k},\vec{r}) = \frac{1}{e^{(\varepsilon_k - eV^{tot}(\vec{r}) - \mu)/k_B T} + 1}$$

$$= f^0(\vec{k}; \mu + eV^{tot}(\vec{r}))$$

where $f^0(\vec{k}; \mu) = \dfrac{1}{e^{(\varepsilon_k - \mu)/k_B T} + 1}$ is the

equilibrium distribution when $V = 0$.

So the effect of $V^{tot}(\vec{r})$ can be viewed as if there is now a spatially varying chemical potential $\mu + eV^{tot}(\vec{r})$ (this is sometimes called the electro-chemical potential)

then

$$\delta\rho(\vec{r}) = -e \int \frac{d^3k}{4\pi^3} \left[f^0(\vec{k}; \mu + eV^{tot}(\vec{r})) - f^0(\vec{k}; \mu) \right]$$

$\uparrow$

includes factor
$\times 2$ for spin degeneracy

expand in small $eV^{tot} \ll \mu \approx \varepsilon_F$

$$\delta \rho(\vec{r}) = -e \int \frac{d^3k}{4\pi^3} \frac{\partial f^0}{\partial \mu} eV^{tot}(\vec{r})$$

$$= -e^2 V^{tot}(\vec{r}) \frac{\partial}{\partial \mu} \underbrace{\int \frac{d^3k}{4\pi^3} f^0(\vec{k})}$$

equilibrium density $n(\mu)$ as function of chemical pot μ.

$$= -e^2 V^{tot}(\vec{r}) \frac{\partial n}{\partial \mu}$$

So also $\quad \delta \rho(\vec{k}) = -e^2 V^{tot}(\vec{k}) \frac{\partial n}{\partial \mu}$

$$\frac{\delta \rho(\vec{k})}{V^{tot}(\vec{k})} = -e^2 \left(\frac{\partial n}{\partial \mu} \right)$$

So dielectric function is

$$\varepsilon(\vec{k}) = 1 - \frac{4\pi}{k^2} \frac{\delta \rho(\vec{k})}{V^{tot}(\vec{k})}$$

$$\boxed{\varepsilon(\vec{k}) = 1 + \frac{4\pi e^2}{k^2} \frac{\partial n}{\partial \mu}}$$

This is called the __Thomas-Fermi__ dielectric function, and it can be written in the form

$$\varepsilon(\vec{k}) = 1 + k_0^2/k^2 \qquad k_0^2 \equiv 4\pi e^2 \frac{\partial n}{\partial \mu}$$

where $1/k_0$ is called the __screening length__

Before considering the physical consequences of $\mathcal{E}(\vec{k})$ as above, lets first compute k_0.

As $T \to 0$, $\mu \to \mathcal{E}_F$ the Fermi energy. Now $\mathcal{E}_F$ is defined by

$$M = \int_0^{\mathcal{E}_F} d\mathcal{E} \, g(\mathcal{E}) \implies \frac{\partial M}{\partial \mathcal{E}_F} = g(\mathcal{E}_F)$$

$\uparrow$ density of states

For a free electron gas, $g(\mathcal{E}_F) = \frac{3}{2} \frac{M}{\mathcal{E}_F}$

So $k_0^2 = 4\pi e^2 g(\mathcal{E}_F) = \frac{6\pi e^2 M}{\mathcal{E}_F}$

~~~~~~

We can compare this to what one would get using classical Boltzmann statistics for the electrons, instead of the quantum Fermi-Dirac statistics. Then, the probability distribution for an electron with momentum $\hbar \vec{k}$ would be

$$f(\vec{k}, \vec{r}) = C e^{-(\mathcal{E}_k - eV^{tot}(\vec{r}))/k_B T}$$

$\uparrow$ normalization constant

expand for small eV^{tot}

$$f(\vec{k}, \vec{r}) \simeq C e^{-\varepsilon_k / k_B T} \left[1 + \frac{eV^{tot}(\vec{r})}{k_B T} \right]$$

$$= f^0(k) \left[1 + \frac{eV^{tot}(\vec{r})}{k_B T} \right]$$

$$\delta\rho(\vec{r}) = -e \int \frac{d^3 k}{4\pi^3} \left[f(\vec{k}, \vec{r}) - f^0(\vec{k}) \right]$$

$$= -e \int \frac{d^3 k}{4\pi^3} \, f^0(k) \, \frac{eV^{tot}(\vec{r})}{k_B T}$$

$$= -e^2 V^{tot}(\vec{r}) \frac{m}{k_B T}$$

So $\quad \dfrac{\delta\rho(\vec{k})}{V^{tot}(\vec{k})} = -\dfrac{e^2 m}{k_B T} \quad$ which gives

$$\varepsilon(\vec{k}) = 1 - \frac{4\pi}{k^2} \frac{\delta\rho(\vec{k})}{V^{tot}(\vec{k})} = 1 + \frac{4\pi e^2}{k^2} \frac{m}{k_B T}$$

This is known as the Debye-Huckel dielectric function — it applies to a liquid or gas of charged particles obeying classical statistics (for example is $T \gg T_F$). It has the same function form as the Thomas-Fermi dielectric function, but now with

$$k_0^2 = \frac{4\pi e^2 m}{k_B T}$$

So to compare Thomas - Fermi with classical
Debye - Huckle

$$\frac{k_0^{TF}}{k_0^{DH}} = \left(\frac{6\pi e^2 m}{k_B T_F} \frac{k_B T}{4\pi e^2 m} \right)^{1/2} = \left(\frac{3}{2} \left(\frac{T}{T_F} \right) \right)^{1/2} << 1$$

so   Debye Huckel screening length $\frac{1}{k_0^{DH}} << \frac{1}{k_0^{TF}}$ Thomas-
Fermi screening length, for the same density n.

Back to Thomas - Fermi

$$k_0^2 = \frac{6\pi e^2 m}{\varepsilon_F}$$ using $n = \frac{k_F^3}{3\pi^2}$, $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$

$$\Rightarrow k_0^2 = \frac{4 m e^2}{\pi \hbar^2} k_F$$ using Bohr radius $a_0 = \frac{\hbar^2}{m e^2}$

$$\Rightarrow k_0^2 = \frac{4}{\pi} k_F / a_0$$ using $\frac{4}{3}\pi r_s^3 = \frac{1}{n}$

$$k_F = \left(\frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s}$$

$$\Rightarrow k_0 = \sqrt{ \frac{4}{\pi} \left(\frac{9\pi}{4} \right)^{1/3} } \frac{1}{\sqrt{r_s a_0}}$$

$$\frac{1.56}{\sqrt{r_s a_0}} = \frac{1.56}{\sqrt{r_s a_0}}$$

$$\frac{1}{a_0 k_0} = \frac{1}{1.56} \sqrt{\frac{r_s}{a_0}}$$ $r_s \sim 3 a_0$ for
most metals

$$\Rightarrow \frac{1}{k_0} \sim a_0 \sim .5 \mathring{A}$$ screening length very small !

Thomas-Fermi dielectric function

$$\varepsilon(k) = 1 + k_0^2/k^2 \qquad, \text{ with } 1/k_0 \sim \overset{\circ}{A}$$

One consequence of this form is something
you have already learned in your EM class.

If we take the limit $k \to 0$, then $\varepsilon \to \infty$.
So a uniform electric field applied to a metal
is completely screened out! $E^{tot} = \dfrac{E}{\varepsilon} \to 0$.
In practice, provided the applied E field
is slowly varying on the length scale $1/k_0 \sim \overset{\circ}{A}$,
it is still screened out because ε is so large
for $k \ll k_0$. This is what you learned in EM
— there can be no static macroscopic
electric field inside a metal.

Another extremely important consequence
of the T-F dielectric function is seen if
we consider the effect on a point charge
Q placed in the electron gas.

The "applied" potential ~~if the~~ from the
point charge is just the bare Coulomb potential

$$V(\vec{r}) = \frac{Q}{r}$$

The Fourier transform of the Coulomb potential is

$$V(\vec{k}) = \frac{4\pi Q}{k^2}$$

$$\Rightarrow V^{tot}(\vec{k}) = \frac{V(\vec{k})}{\varepsilon(\vec{k})} = \frac{4\pi Q}{k^2} \frac{1}{1 + k_0^2/k^2}$$

$$= \frac{4\pi Q}{k_0^2 + k^2}$$

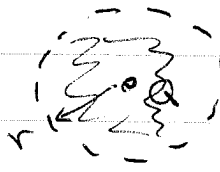whereas $V(\vec{k}) \to \infty$ as $k \to 0$, indicating a <u>long</u> ranged interaction, $V^{tot}(\vec{k}) \to const$ as $k \to \infty$, indicating a short ranged interaction.

If we Fourier transform $V^{tot}(\vec{k})$ back to real space we get the interaction

$$V^{tot}(\vec{r}) = \frac{Q e^{-k_0 r}}{r}$$

called the "Yukawa potential" or the "screened Coulomb potential"

The effect of the dielectric function due to the free electrons is to "screen" the long range Coulomb potential so it looks short ranged with an interaction length $= 1/k_0$. On length scales $r \gg 1/k_0$, the effect of the charge Q is entirely negligible. We say that the electrons have screened out the charge Q.

Physically, what is going on is as follows:

electrons get attracted to charge Q and so the average electron density about Q increases above average. This cloud of electron charge at Q "screens" the charge Q. If one computes the total charge (ie Q + induced electron charge) in a sphere of radius r centered on Q, this total charge decreases to zero as $r \to \infty$. Decay of this total charge is on length scale $1/k_0$.

Compare this to behavior in a "dielectric" (ie an insulator) from EM class. If you put a point charge Q in a dielectric, it polarizes the material creating bound charges at Q so that the total charge at Q becomes $Q + Q_{bound} = Q/\varepsilon$, where ε is the finite "dielectric constant". A metal is like a dielectric with an infinite dielectric constant so that $Q/\varepsilon \to 0$! (any free charge in a metal must lie on its surface!). The dependence of ε on wave vector k, describes how the metal screens charges on finite length scales - so $Q/\varepsilon \to 0$ is really just a statement about the $r \to \infty$, or $k \to 0$ limit of the metal.

This behavior due to $\varepsilon(k)$ explains why it is not necessarily such a bad approximation that we have ignored electron-ion interactions and electron-electron interactions. Instead of these being strong long ranged Coulomb interactions, the screening by $\varepsilon(k)$ converts them into very short ranged interactions — interaction range is $1/k_0 \sim \AA$.

When we later consider the effect of the periodic potential of the ions on the electrons, we will see that in many cases we can describe its effect as a weak perturbation on plane wave electron eigenstates. That the electron-ion interaction can be treated as a weak perturbation is due to the screening of the ion potential by $\varepsilon(\vec{k})$.

When we think about the electron-electron interaction between a given pair of electrons, we can similarly think of this as being screened by all the other electrons. Hence it too is short ranged and not as important as one might naively have expected. There is one complication that arises when one thinks

about the screening of the electron-electron interaction. Since all electrons are identical, we cannot really distinguish between a given pair of interacting electrons and the other electrons that are screening this interaction. But despite this complication, the idea that $\varepsilon(k)$ ~~screen~~ screens the e-e interaction and makes it short ranged, remains essentially correct. See Ashcroft & Mermin Chpt 17 or Kittel Chpt 14 for further discussion.