

# Motion in Uniform Magnetic field

$$\dot{\vec{r}} = \frac{1}{\hbar} \frac{\partial \mathcal{E}}{\partial \vec{k}} \quad \hbar \dot{\vec{k}} = -e \frac{i}{c} \vec{v}(\vec{k}) \times \vec{H}$$

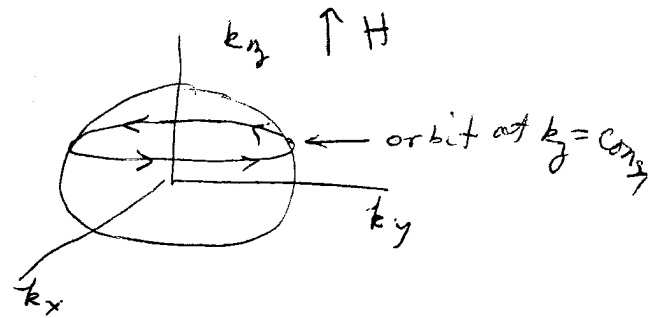
For motion in uniform field,  $\dot{\mathcal{E}}(\vec{k}(t)) = \frac{d\mathcal{E}}{d\vec{k}} \cdot \frac{d\vec{k}}{dt} = \hbar \vec{v} \cdot \dot{\vec{k}} = 0$

since  $\vec{v} \cdot (\vec{v} \times \vec{H}) = 0$

so electron moves on surface of constant energy,  
 also  $\frac{d}{dt}(\vec{k} \cdot \vec{H}) = \dot{\vec{k}} \cdot \vec{H} = 0$  as  $\vec{H} \cdot (\vec{v} \times \vec{H}) = 0$

⇒ electrons move on curves formed by intersection of plane of constant  $k_z$  (take  $\vec{H}$  in  $z$  dir  $k_{||}$ , with surfaces of constant energy.

For spherical energy surface

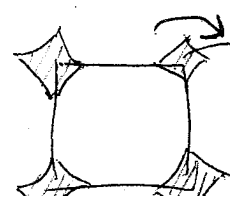
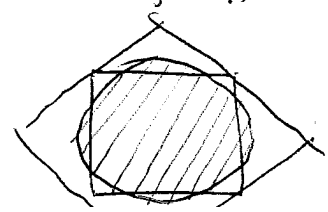


Sence of orbit: since  $\vec{v} = \frac{1}{\hbar} \frac{\partial \mathcal{E}}{\partial \vec{k}}$  points from low  $\mathcal{E}$  to higher  $\mathcal{E}$ .  
 If  $\vec{H}$  is up, one walks in orbit so that higher energy states are on right as  $\dot{\vec{k}} \sim \vec{H} \times \vec{v}$

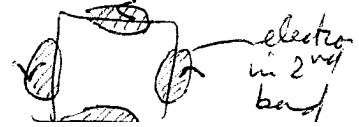
closed orbits, If surface encloses region of higher energy, direction is opposite than if surface encloses lower energy (electron orbit) (hole orbit).

ex: 3-d cubic,  $\vec{H} \parallel \hat{z}$  so in nearly free electron

approx



holes in 1st band



electron in 2nd band

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{km} - \delta_{im} \delta_{kl}$$

The real space orbits ( $\vec{r}(t)$ ) can be found:

$$\vec{r}_{\perp} \equiv \vec{r} - \hat{H}(\hat{H} \cdot \vec{r}) \quad \text{position in plane } \perp \text{ to } \hat{H}$$

$$\begin{aligned} \hat{H} \times \hbar \dot{\vec{r}} &= -\frac{e\hbar}{c} \hat{H} \times (\vec{v} \times \hat{H}) = -\frac{e}{c} H (\vec{r} - \hat{H}(\hat{H} \cdot \vec{r})) \\ &= -\frac{eH}{c} \vec{r}_{\perp} \end{aligned}$$

using  $\vec{v} = \dot{\vec{r}}$   
& vector identity

$$\text{so } \vec{r}_{\perp}(t) - \vec{r}_{\perp}(0) = -\frac{\hbar c}{eH} \hat{H} \times (\vec{k}(t) - \vec{k}(0))$$

So  $\vec{r}_{\perp}$  orbit is just  $\vec{k}$  orbit rotated by  $90^\circ$  about  $\hat{H}$ ,  
and scaled by  $\frac{\hbar c}{eH}$  *clockwise*

in // direction

$$r_{\parallel}(t) = r_{\parallel}(0) + \int_0^t v_{\parallel}(t) dt = r_{\parallel}(0) + \int_0^t \frac{1}{\hbar} \frac{\partial \epsilon}{\partial k_{\parallel}} dt$$

need not be uniform in  $t$  as  $\frac{\partial \epsilon}{\partial k_{\parallel}}(\vec{k})$  can vary  
as  $k_{\perp}$  varies.

For spherical energy surface, we get classical result:  
electron moves in circular orbit  $\perp$  to  $\hat{H}$ .

However energy surfaces need not be spherical  
- (when they get too near zone boundaries) - need  
not be closed curves! see figure 12.8 in text

~~When orbits are open, applying  $\hat{H}$  can lead to~~

Motion in uniform  $\perp$   $\vec{E}$  and  $\vec{H}$  fields  
Hall effect and magnetoresistance

$$\hbar \dot{\vec{k}} = -e \left[ \vec{E} + \frac{\vec{v}}{c}(\vec{k}) \times \vec{H} \right]$$

$$\Rightarrow \hat{H} \times \hbar \dot{\vec{k}} = -e \hat{H} \times \vec{E} - \frac{eH}{c} \dot{\vec{r}}_{\perp}$$

$$\dot{\vec{r}}_{\perp} = -\frac{\hbar c}{eH} \hat{H} \times \dot{\vec{k}} + \vec{w} \quad \vec{w} = \frac{cE}{H} (\hat{E} \times \hat{H})$$

Motion is as before, but with drift velocity  $\vec{w}$  added.

To determine orbits in  $k$  space note:

$$\hbar \dot{\vec{k}} = -e \vec{E} - \frac{e}{c} \frac{\perp}{\hbar} \frac{\partial \epsilon}{\partial \vec{k}} \times \vec{H}$$

write  $\vec{E} = -(\vec{E} \times \hat{H}) \times \hat{H}$   
true when  $\vec{E} \perp \vec{H}$

$$= -\frac{e}{c\hbar} \left( \frac{\partial \epsilon}{\partial \vec{k}} - \frac{c\hbar E}{H} \hat{E} \times \hat{H} \right) \times \vec{H}$$

$$\equiv -\frac{e}{c\hbar} \frac{\partial \bar{\epsilon}}{\partial \vec{k}} \times \vec{H} \quad \bar{\epsilon} = \epsilon - \hbar \vec{k} \cdot \vec{w}$$

Same as if  $\vec{E}$  was absent and band structure replaced by  $\bar{\epsilon}(\vec{k}) = \epsilon(\vec{k}) - \hbar \vec{k} \cdot \vec{w}$

Orbits are intersections of surfaces of constant  $\bar{\epsilon}$  with planes  $\perp$  to  $\vec{H}$

We will assume that  $-\hbar \vec{k} \cdot \vec{w}$  small enough so that if the constant  $\epsilon(\vec{k})$  surface is closed (open) so is the constant  $\bar{\epsilon}(\vec{k})$  surface. Good approx in most cases - see text for estimate of numbers.

in nearly free electron model

$$E(\vec{k}) \approx \frac{\hbar^2 k^2}{2m}$$

surface of constant energy  $E$   
is sphere of radius

$$\sqrt{\frac{2mE}{\hbar^2}} = k \quad \text{in } k\text{-space}$$

$$\bar{E}(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} - \hbar \vec{w} \cdot \vec{k} \quad \text{surface of constant } \bar{E}$$

is given by

$$\frac{\hbar^2}{2m} \left| \vec{k} - \frac{m\vec{w}}{\hbar} \right|^2 = \bar{E} + \frac{1}{2}m\vec{w}^2$$

sphere in  $k$ -space of radius

$$k = \sqrt{\frac{2m}{\hbar^2} \left( \bar{E} + \frac{1}{2}m\vec{w}^2 \right)}$$

centered about  $\vec{k}_0 = m\vec{w}/\hbar$

surface of constant  $\bar{E}$  is  
shifted by  $\vec{w} \cdot \vec{k}$  term in direction  
 $\vec{w}$

Hall effect:  $\dot{\vec{r}}_{\perp} = -\frac{tc}{eH} \hat{H} \times \dot{\vec{k}} + \vec{w}$ ,  $\vec{w} = \frac{cE}{H} (\hat{E} \times \hat{H})$

current in plane  $\perp$  to  $H$  is

$\vec{j} = -ne\langle \dot{\vec{r}}_{\perp} \rangle$  where  $\langle \dot{\vec{r}}_{\perp} \rangle$  is steady state average over all occupied

$\vec{j} = -ne\vec{w} + \frac{netc}{eH} \hat{H} \times \langle \dot{\vec{k}} \rangle$  electron orbits, and over collisions

Case (1) All occupied (or unoccupied) orbits are closed

Then for large enough  $H$  so that  $\omega_c \tau \gg 1$  (where  $\tau$  is collision time, and  $\omega_c = eH/m^*c$ ), electron makes many periods of its closed orbits between successive collisions.

We can estimate  $\langle \dot{\vec{k}} \rangle$  in this large  $H$  case as follows: Averaging over electron motion between two successive collisions at  $t=0$  and  $t=t_0$  we get

$$\langle \dot{\vec{k}} \rangle = \frac{1}{t_0} \int_0^{t_0} \dot{\vec{k}}(t) dt = \frac{\vec{k}(t_0) - \vec{k}(0)}{t_0}$$

where  $\vec{k}(0)$  is wave vector of electron as it emerges from the first collision at  $t=0$ , and  $\vec{k}(t_0)$  is wave vector of electron just before second collision at  $t=t_0$

As in the Drude model, we may assume that electrons emerge from a collision with an equilibrium distribution determined by the local temperature + chemical potential.

Since the Fermi distribution  $f(\epsilon(\vec{k})) = \frac{1}{1 + e^{\beta(\epsilon - \mu)}}$  depends on  $\vec{k}$  only via energy  $\epsilon(\vec{k})$ , and  $\epsilon(\vec{k}) = \epsilon(-\vec{k})$ .

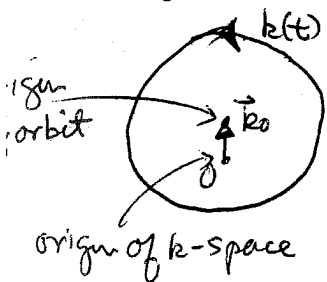
We have, after averaging over the electron emerging from the collision at  $t=0$ ,  $\langle \vec{k}(0) \rangle = 0$ . So  $\langle \vec{k} \rangle = \vec{k}(t_0)/t_0$ .

We now average over the time until the second collision,  $\langle t_0 \rangle = \tau$  (this time is distributed randomly with average equal to  $\tau$ ). Since  $\omega_c \tau \gg 1$ , ~~all~~ the electron makes many orbits between collisions,  $\Rightarrow \vec{k}(t_0)$  when averaged over collision time  $t_0$ , is equally likely to lie anywhere along the closed orbit.

$\Rightarrow \langle \vec{k}(t_0) \rangle = (\text{average } \vec{k} \text{ on orbit})$ . If electric field  $\vec{E} = 0$ , then (average  $\vec{k}$  on orbit)  $= 0$  also. But when  $E \neq 0$ , (average  $\vec{k}$  on orbit)  $\sim m^* \vec{w} / \hbar$ . To see this, use effective mass approximation,  $\epsilon(\vec{k}) \approx \frac{\hbar^2 k^2}{2m^*}$ .

then orbit lies on curve of constant

$\bar{\epsilon}(\vec{k}) = \epsilon(\vec{k}) - \hbar \vec{k} \cdot \vec{w}$ , which lies on sphere centered at  $\vec{k}_0 = m^* \vec{w} / \hbar$ . So (average  $\vec{k}$  on orbit)  $= \langle \vec{k}(t_0) \rangle = \vec{k}_0$



$$\Rightarrow \langle \vec{k} \rangle = \frac{\langle \vec{k}(t_0) \rangle}{\tau} = \frac{\vec{k}_0}{\tau} = \frac{m^* \vec{w}}{\hbar \tau}$$

So contribution of  $\langle \vec{k} \rangle$  term to current is

$$\frac{ne\hbar c}{eH} \hat{H} \times \frac{m^* \vec{w}}{\hbar \tau} = \frac{ne}{\omega_c \tau} \hat{H} \times \vec{w}$$

smaller than drift contribution to current  $\vec{j} \approx -ne\vec{w}$  by a factor  $\frac{1}{\omega_c \tau} \ll 1$

So  $\vec{j} \approx -ne\vec{w}$  given just by drift velocity  $\vec{w}$  in high field limit.

Let us keep the contribution from  $\langle \vec{k} \rangle$ .  
 Then (we will need that term to get magnetoresistance)

$$\vec{j} = -ne\vec{w} + \frac{ne}{\omega_c\tau} \hat{H} \times \vec{w} \quad \vec{w} = \frac{eE}{H} (\hat{E} \times \hat{H})$$

Take  $\hat{H} = \hat{z}$

$$\vec{j} = \frac{neE}{H} \left( \hat{z} \times \vec{E} + \frac{1}{\omega_c\tau} \vec{E} \right)$$

write in terms of a conductivity tensor  $\vec{j} = \vec{\sigma} \cdot \vec{E}$

$$\vec{\sigma} = \frac{neE}{H} \begin{pmatrix} \frac{1}{\omega_c\tau} & -1 \\ 1 & \frac{1}{\omega_c\tau} \end{pmatrix}$$

or using  $\frac{neE}{H} = \left( \frac{ne^2\tau}{m^*} \right) \left( \frac{m^*c}{eH\tau} \right) = \frac{\sigma_0}{\omega_c\tau}$

$$\vec{\sigma} = \sigma_0 \begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & -1 \\ \frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$$

Resistivity tensor is then

$$\vec{\rho} = \vec{\sigma}^{-1} = \frac{1/\sigma_0}{\frac{1}{(\omega_c\tau)^4} + \frac{1}{(\omega_c\tau)^2}} \begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & \frac{1}{\omega_c\tau} \\ -\frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$$

$$= \frac{1/\sigma_0}{1 + \frac{1}{(\omega_c\tau)^2}} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}$$