PHY 103: Scales and Musical Temperament

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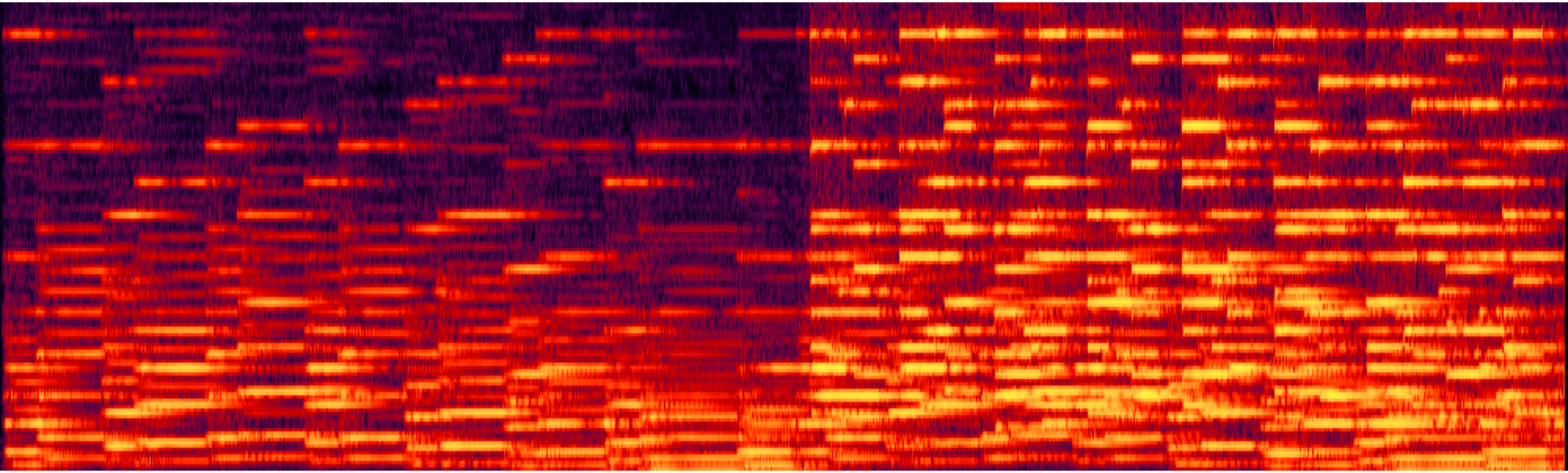
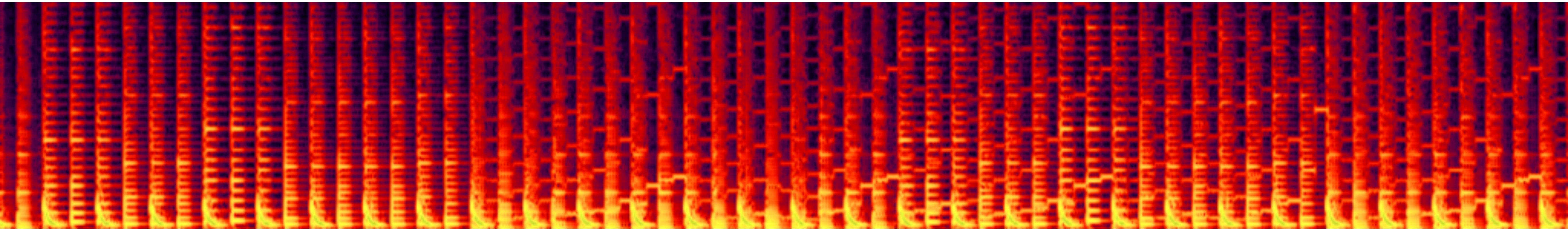
Musical Structure

- We've talked a lot about the physics of producing sounds in instruments
- We can build instruments to play any fundamental tone and overtone series

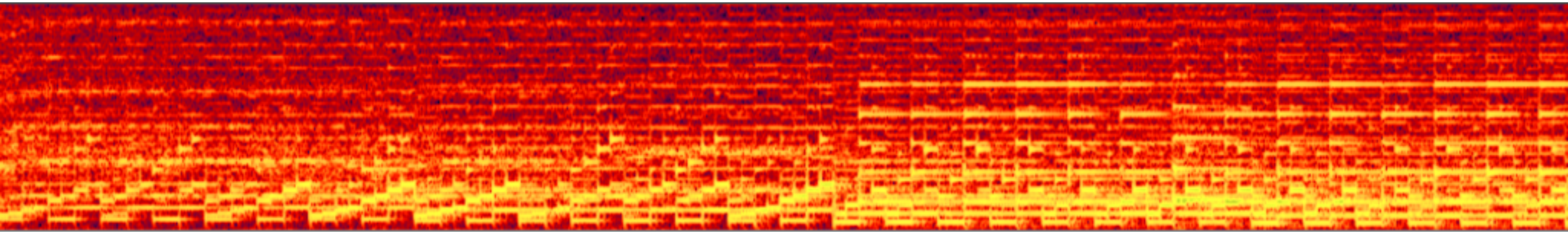


Scales

- In practice, we don't do that. There are agreed-upon conventions for how notes are supposed to sound



- Why is that? How did those conventions come about? Is there a reason for it?



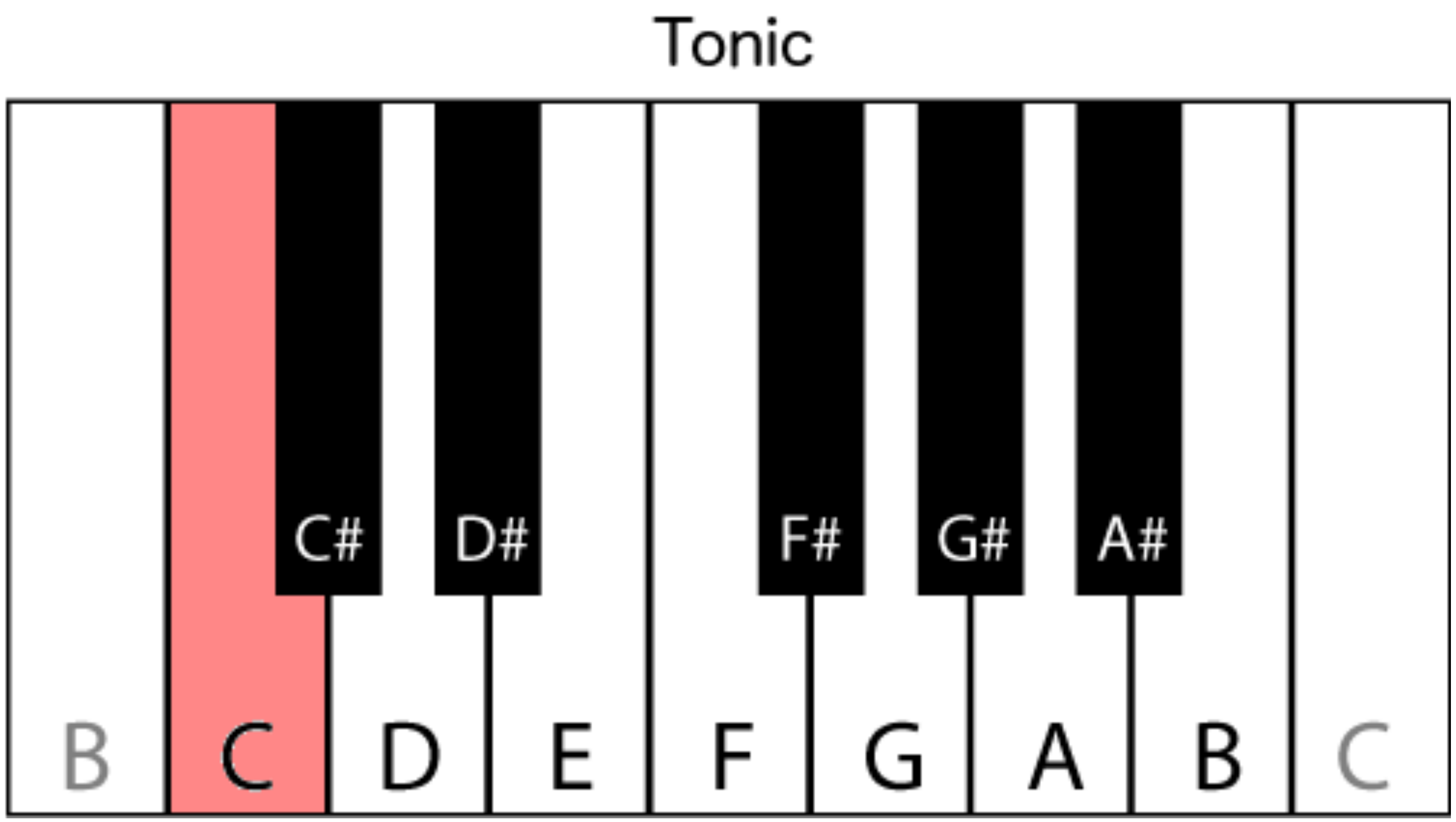
What's a Scale?

- A scale is a pattern of notes, usually within an octave
- In Western music we use the diatonic scale

- This scale contains seven distinct pitch classes and is part of a general class of scales known as *heptatonic*
- Doubling the frequency of a tone in this scale requires going up by 8 notes: hence the term "octave"

Basic Terminology

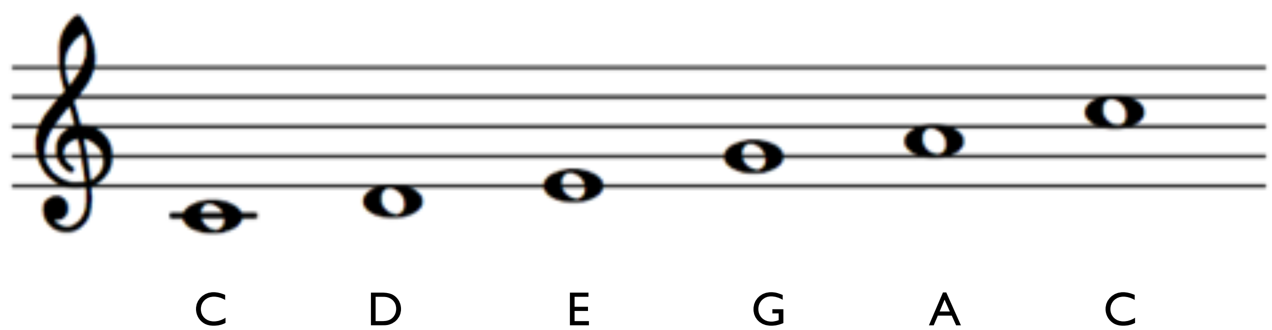
- Note combinations are described with respect to their position in the scale



- Ignore the modifiers "major" and "perfect" for now; we'll come back to those in a few minutes

Pentatonic Scale

- The diatonic scale (and other heptatonic scales) are found all over the world, but are not universal
- Traditional Asian music is based on the pentatonic scale:



- Familiar example of the pentatonic scale: opening of "Oh! Susanna" by Stephen Collins Foster

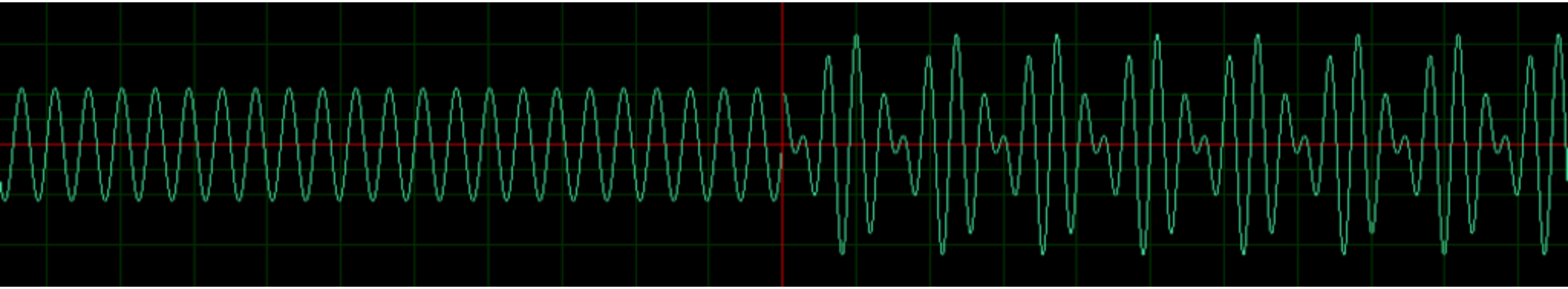
Origins of the Scale

- Where do these note patterns come from?
- Why can they be found around the world?

Neolithic bone flutes (7000 BC), Jiahu, China

Psychology of Hearing

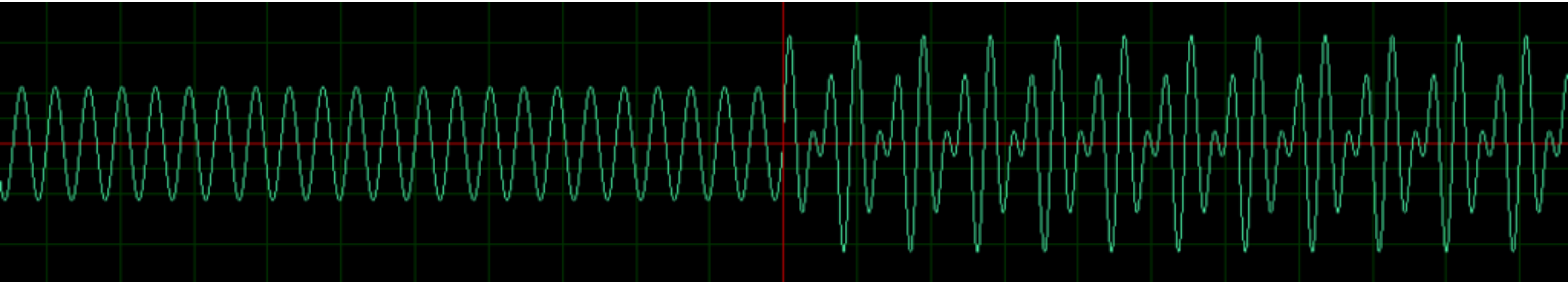
- Our ears interpret musical intervals in terms of ratios
- Perfect 4th: interval between two pitches whose fundamental frequencies form the ratio 4:3
    - Ex: **A4** (440 Hz), **D5** (586.67 Hz = 4/3 × 440 Hz)



- Tone sample: 2 s of **A4**, then 2 s of **A4** + **D5**

Psychology of Hearing

- Our ears interpret musical intervals in terms of ratios
- Perfect 5th: interval between two pitches whose fundamental frequencies form the ratio 3:2
  - Ex: **A4** (440 Hz), **E5** (660 Hz = 3/2 × 440 Hz)



  - Tone sample: 2 s of **A4**, then 2 s of **A4** + **E5**

Chromatic Scale

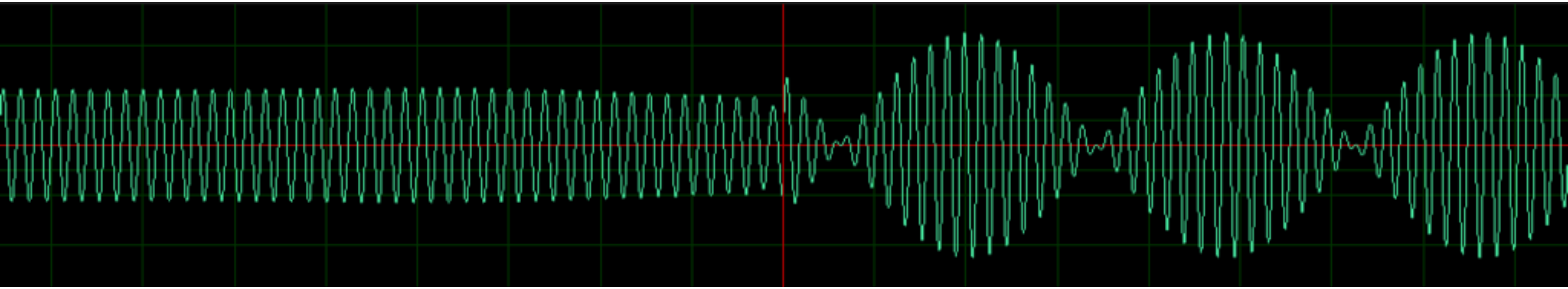
- The complete scale in an octave is the 12-pitch chromatic scale

- The 12 pitches are separated by half notes, or semitones, the smallest musical interval used in Western music

- Semitones are quite dissonant when sounded harmonically; i.e., they don't sound pleasant

Dissonant Semitones

- Why do semitones sound so dissonant?
- Look at the waveform produced by playing a pure middle C (261.63 Hz) and C# (277.19 Hz)



- Significant beating is present; our ears don't like it
- Question: what's the beat frequency?

Origin of Scales and Pitch

- We like octaves, which are a frequency ratio of 2:1
- We also like the sound of small integer ratios of frequency, e.g., the perfect 5th (3:2)
- These kinds of integer ratios tend to show up when you tune an instrument by ear, because they "sound right." Our ears like it when harmonics align due to the lack of dissonant beats
- However, the manner in which notes are scaled within an octave can be pretty arbitrary
- Even the pitch of notes has evolved over time

Absolute Pitch

- In the baroque period, A = 415 Hz was the standard

- Bach used 415 Hz. Handel used 422.5 Hz. The international tuning standard of A4 = 440 Hz was widely adopted around 1920

Building Scales

Pythagorean Tuning
Just Temperament
Equal Temperament

Overtones of the String

Berg and Stork, Ch. 3

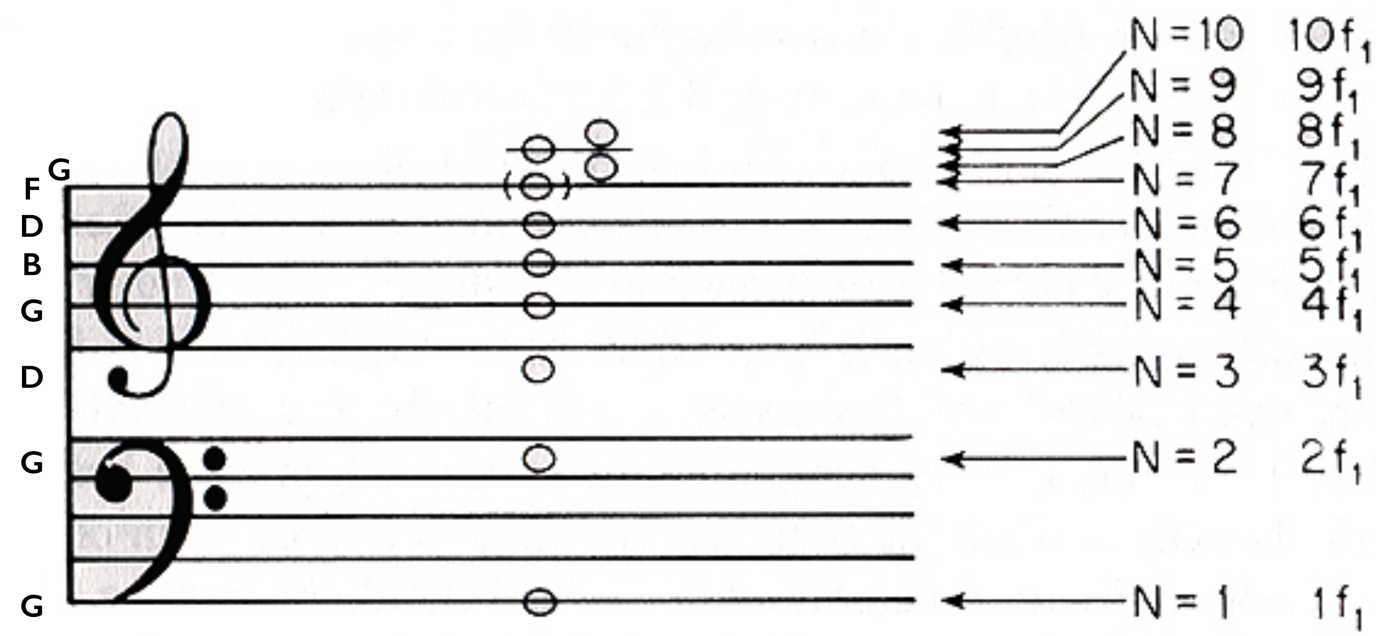


Figure 3–12 Notes on the musical staff having frequencies closest to the notes in the overtone series of G_2; each note is labeled by its harmonic number and the frequency of the corresponding harmonic.

TABLE 3–3 MUSICAL INTERVALS BETWEEN THE FUNDAMENTAL AND OTHER NOTES OF THE OVERTONE SERIES

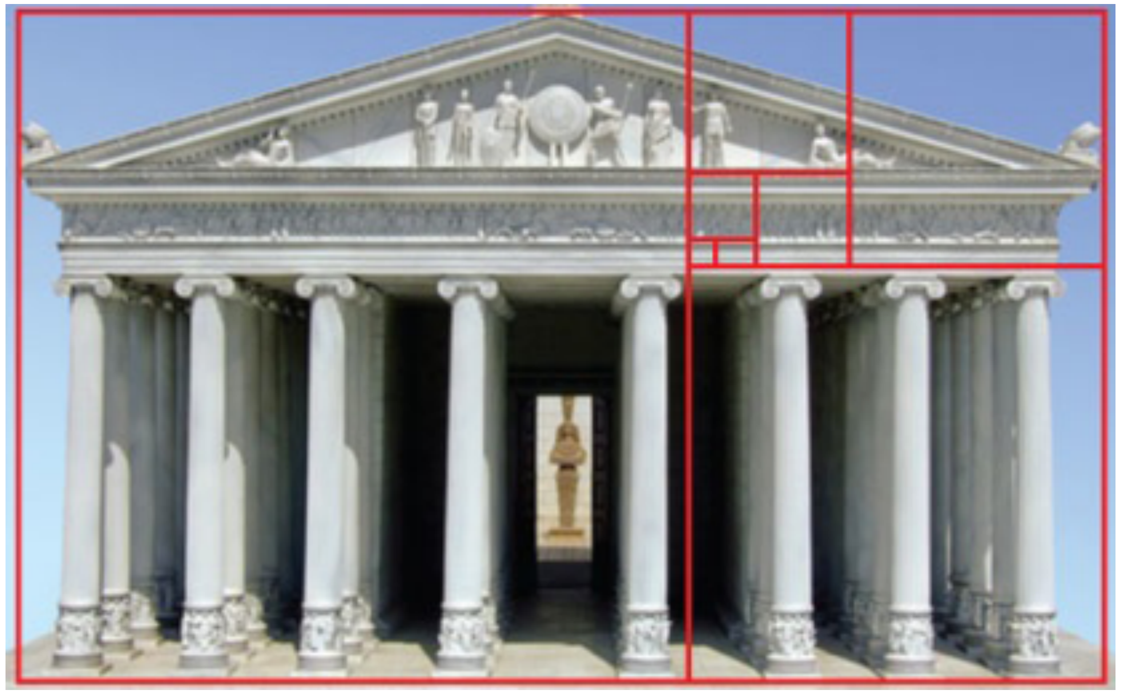
N	f	Interval
1	f_1	Unison
2	$2f_1$	One octave
3	$3f_1$	One octave + one perfect fifth
4	$4f_1$	Two octaves
5	$5f_1$	Two octaves + one major third
6	$6f_1$	Two octaves + one perfect fifth
7	$7f_1$	Two octaves + one minor seventh
8	$8f_1$	Three octaves

Origin of the Diatonic Scale

- Start with the tonic of the string (f) and go *up* by an octave: you get a doubling of the frequency ($2f$)
- An octave + 5th gives the string's third harmonic ($3f$)
- Drop the octave + 5th note *down* by one octave to the first 5th and the frequency is divided by two ($3f/2$)
- Hence, the perfect fifth is in a 3:2 ratio with the tonic
- Pythagorean tuning: build up the chromatic scale of 12 notes by climbing up the scale by 5ths and down by octaves

Pythagorean Tuning

- The Pythagorean tuning system is one of the first theoretical tuning systems in Western music (that we know about)



- The Pythagorean scale appeals to symmetry: you can construct the chromatic scale in terms of simple integer ratios of a fundamental frequency

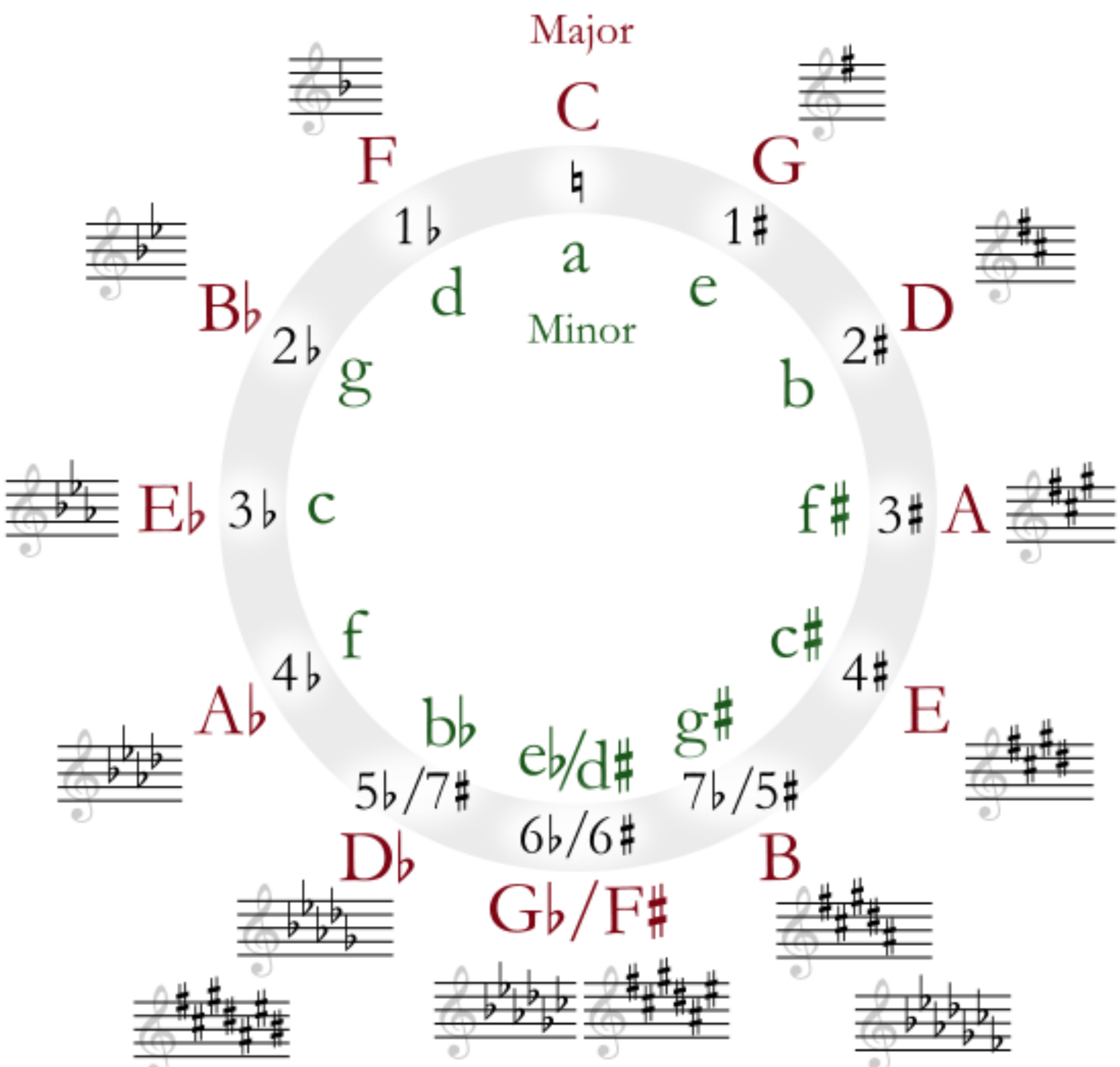
Pythagorean Temperament

- Go up by 5ths and down by 8ves to fill the scale:

Berg and Stork, Ch. 9

Note	*Frequency Relation to Tonic*	*Ratio*
C_4	tonic (1.000)	**1.0000**
G_4	$3/2 \cdot C_4$	**1.5000**
$D_5 \rightarrow D_4$	$1/2 \cdot D_5 = 1/2 \cdot (3/2 \cdot G_4) = 1/2 \cdot (3/2 \cdot (3/2 \cdot C_4) = 9/8 \cdot C_4$	**1.1250**
A_4	$3/2 \cdot D_4 = 27/16 \cdot C_4$	**1.6875**
$E_5 \rightarrow E_4$	$1/2 \cdot E_5 = 1/2 \cdot (3/2 \cdot A_4) = 81/64 \cdot C_4$	**1.2656**
B_4	$3/2 \cdot E_4 = 3/2 \cdot (81/64 \cdot C_4) = 243/128 \cdot C_4$	**1.8984**
$B_3 \rightarrow F_4\#$	$3/2 \cdot (1/2 \cdot B_4) = 3/4 \cdot (243/128 \cdot C_4) = 729/512 \cdot C_4$	**1.4238**
…	**…**	**…**

Circle of Fifths



- Another visualization of the construction of the chromatic scale
- Gives the major and minor keys of the 12 pitches

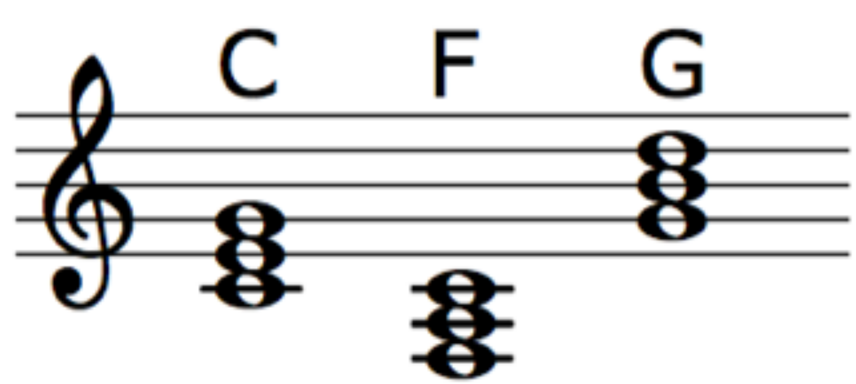
Circle of 5ths in Composition

- The circle of fifths is featured in *Take a Bow* by Muse
- Starts in the key of D, then goes to G, C, F, etc.

- Dramatic political song, lyrics are good but not quite safe for work, so we'll play an instrumental version…

Issues with Pythagorean Scale

- The Pythagorean tuning system is pretty elegant
- Given just a tonic and the 3:2 perfect 5th and 2:1 octave ratios, we can construct the frequencies of all 12 notes in the chromatic scale
- Unfortunately, the major third (81:64) and minor third (32:27) are pretty dissonant in this system
  - No triads!
  - No chords!



The Wolf Interval

- If you go up by a 5th 12 times, you expect to be 7 octaves above the starting point
- But $(3/2)^{12} \cong 129.74$, and $2^7 = 128$; so the circle of fifths doesn't fully close in Pythagorean tuning
- One of the 5th intervals must not match the prescribed frequency ratio. Therefore, there is a dissonant beat (the interval howls like a wolf)

Berg and Stork, Ch. 9

Perfect 5ths and 3rds

- Perfect 5th with frequency ratio 3:2
- Wolf 5th (between C# and A ♭), frequency ratio is

  1:218/311=1.4798
- Perfect 3rd with frequency ratio 5:4
- Pythagorean 3rd with frequency ratio 1.2656

- Sawtooth waves used to include overtones and make the dissonance in Pythagorean tuning more obvious

Just Temperament

- A tuning system in which all the frequencies in an octave are related by very simple integer ratios is said to use just intonation or just temperament
- Frequency ratios obtained in the most basic form of a major scale, relative to the tonic, when using just intonation, are:
  - 1:1, 9:8, 5:4, 4:3, 3:2, 5:3, 15:8, 2:1
- Can get this by tuning the 3rds and allowing some of the 5ths to be slightly out of tune
- Chords no longer dissonant; richer music is possible

Issues with Just Temperament

- Unfortunately, simple frequency ratios don't solve the problems of dissonant chords
- As in the Pythagorean system, there are certain keys and chords that are unplayable in just temperament
- With an integer frequency ratio, tuning errors have to accumulate in certain chords
- Changing keys is also tricky; you have to be careful about how frequencies are calculated

Equal Temperament

- Equal temperament is an attempt to get away from the problem of errors showing up in certain chords
- Idea: tuning errors are distributed equally over all possible triads
- All triads become equal, making key changes much, much easier
- Cost: mild dissonance is present in many chords

12 Tone Equal Temperament

- We want to fit 12 tones equally with an octave, i.e., between frequencies f and $2f$
- How to do it?
- Could try to space the frequencies evenly, in other words, $f_n = (1+n/12)f$

Note	C	C#	D	D#	E	F	F#	G	G#	A	B ♭	B	C
n	0	1	2	3	4	5	6	7	8	9	10	11	12
Ratio	1.00	1.083	1.167	1.25	1.333	1.417	1.5	1.583	1.667	1.75	1.833	1.917	2

- Problem: the diatonic scale sounds *just awful*

12 Tone Equal Temperament

- Better: use a multiplicative factor such that $f_n = a^{n/12} f$
- For $f_{12} = 2f$ (one octave) we need $a = 2$. Therefore,

Note	C	C#	D	D#	**E**	**F**	F#	**G**	G#	A	B♭	B	C
n	0	1	2	3	**4**	**5**	6	**7**	8	9	10	11	12
Ratio	1.00	1.0595	1.122	1.189	**1.26**	**1.335**	1.414	**1.498**	1.587	1.682	1.782	1.888	2

Major 3rd:
5/4 = 1.25

Perfect 4th:
4/3 = 1.333

Perfect 5th:
3/2 = 1.5

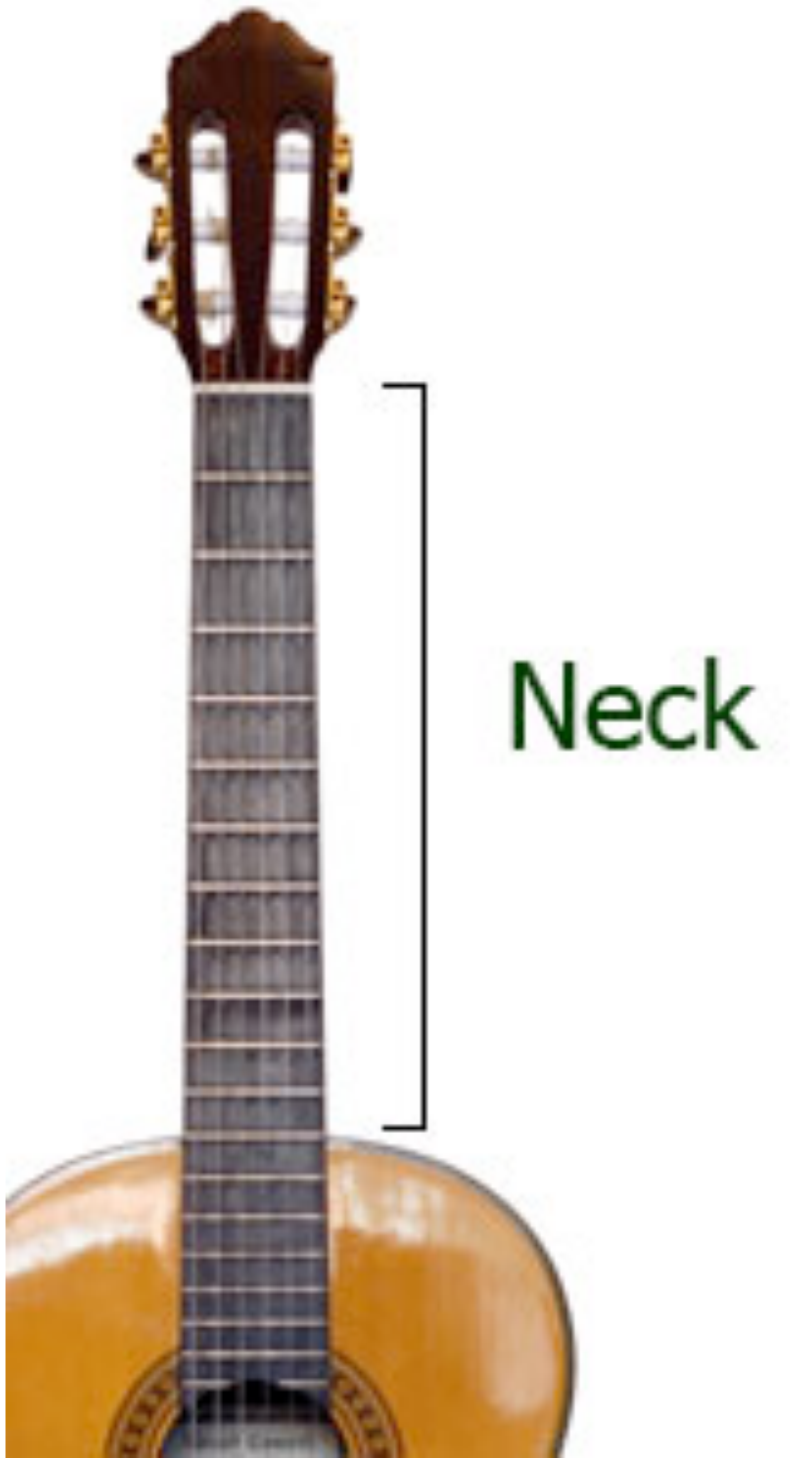
- Diatonic scale sounds pretty good!

Logarithmic Scale

- The equal-tempered scale is not equally spaced in units of *f*; it is equally spaced in units of log *f*
- Example: observe increase in log *f* per semitone

Note	C	C#	D	D#	E	F	F#	G	G#	A	B♭	B	C
n	0	1	2	3	4	5	6	7	8	9	10	11	12
f_n	1.00	1.0595	1.122	1.189	1.26	1.335	1.414	1.498	1.587	1.682	1.782	1.888	2
$\log f_n$	0.000	0.025	0.05	0.075	0.1	0.125	0.151	0.176	0.201	0.226	0.251	0.276	0.301
$\log f_n/f_{n-1}$		**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**

Logarithmic Scale



- The logarithmic scale shows up in instrument design
- Notice how the guitar frets get closer together as you move down the neck
- They are equally spaced by the same multiplicative factor $2^{1/12} \cong 1.0595$
- Equal temperament makes design easy, as long as you remember this factor

Musical Cents

- Cents are a subdivision of the semitone that you will use for tuning your instruments
- Intervals between notes are described using cents
- The definition is: $¢ = 1200 \log_2(f_2 / f_1)$
- Alternatively: $f_2 / f_1 = 2^{¢/1200}$
- If f_2 is one octave higher than f_1, then $f_2 = 2f_1$, and therefore $¢ = 1200$
- I.e., there are 1200 cents per octave, and 100 cents per semitone

Intervals in Base-10

- If you don't like working in base-2 logarithms, you can convert to base-10
- Note: $1200 \cdot \log_2(f_2/f_1) \cong 3986 \cdot \log(f_2/f_1)$
- Alternatively, $f_2 = f_1 \cdot 2^{n/1200} \cong f_1 \cdot 10^{n/3986}$

Notes + Cents → Frequency

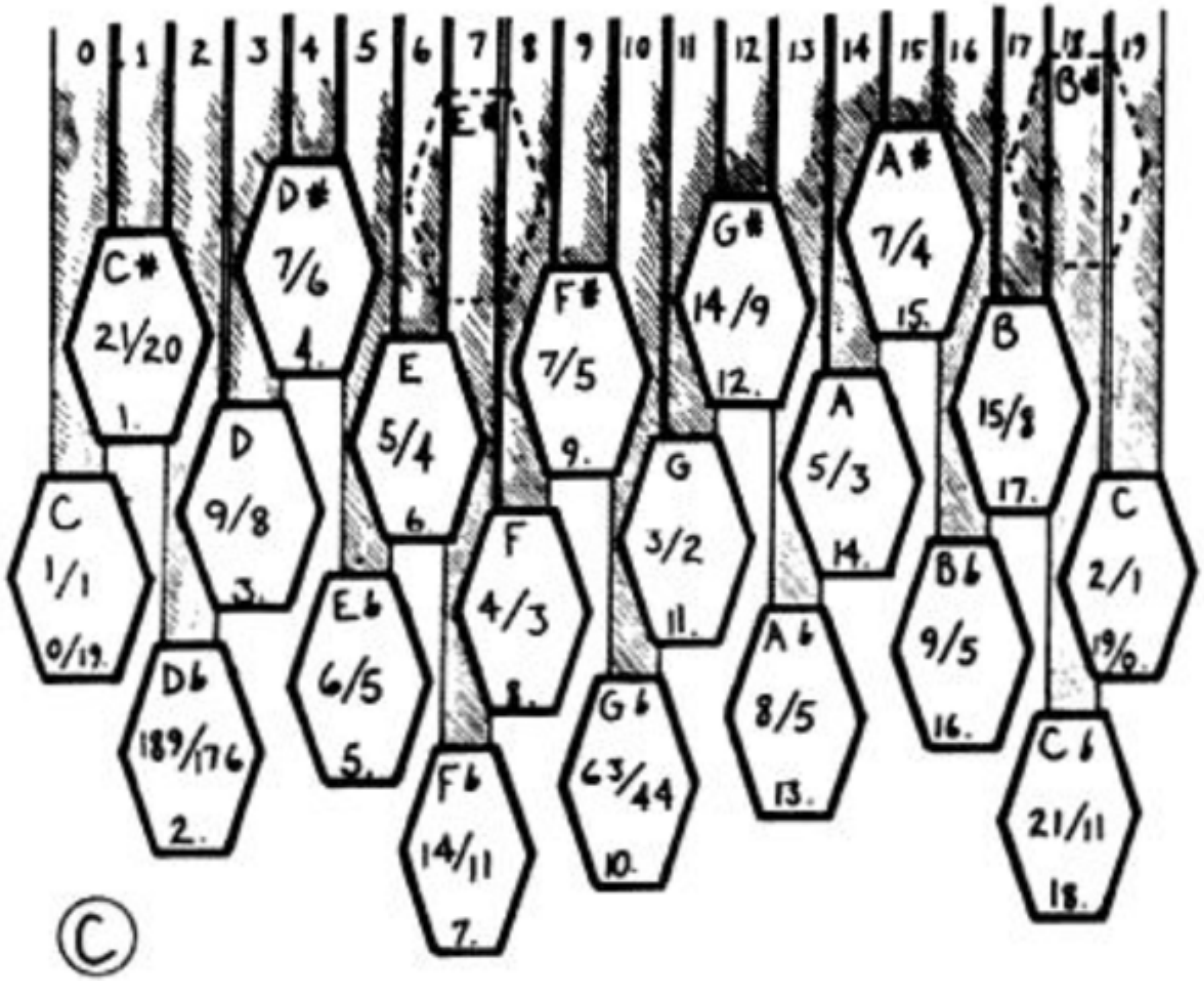
- How to use the digital tuner: what is C4# + 25¢?
  - C4# = 277.18 Hz
  - $2^{25/1200} = 1.0145453$
  - C4# + 25¢ = $2^{25/1200} \times 277.18$ Hz = 281.21 Hz
- What is C4# - 25¢?

Other Equal-Tempered Scales

- 12-tone equal temperament is special in that it is the smallest division of the octave that does a reasonable job of approximating the just intervals we like to hear
- But it's not the only scale that makes a good approximation. Others include:
  - 19-tone scale
  - 24-tone scale (a.k.a. quarter-tone scale)
  - 31-tone scale
  - 53-tone scale

19-Tone Keyboard Layouts

- Proposed layout for 19-tone keyboards (from Hopkin, Ch. 3)



Summary

- The intervals common to music around the world (octaves, 5ths, 3rds, …) are based on simple integer ratios of frequencies
- We like these ratios because they are consonant; we dislike high-integer ratios because they sound dissonant, due to beats
- Just intonation: scale you get when tuning by ear
- Equal temperament: equally spaced semitones on a logarithmic scale
- Reading: Hopkin Ch. 3, Berg and Stork Ch. 9