PHY 103: Fourier Analysis and Waveform Sampling

Segev BenZvi
Department of Physics and Astronomy
University of Rochester

Today's Class

- Topics
  - Fourier's Theorem
  - Nyquist-Shannon Sampling Theorem
  - Nyquist Limit
- Reading
  - Hopkin Ch. 1
  - Berg and Stork Ch. 4

Guess the Song!

- Identify this piece of music…
- If you can't guess (I couldn't), try to guess what era this song comes from
- How can you tell?

10cc: *I'm Not in Love* (1975)

- Here is the first verse of the song…

- Growing up, I heard this on AM radio ("oldies") and FM stations with the 60s/70s/80s format

Fender Rhodes Piano

- The synthesized keyboard gives away the era when this song was written

vintagevibekeyboards (youtube)

commons.mediawiki.org

- It's called a Rhodes (or Fender Rhodes) piano. Very common in pop music from the 1960s to the 1980s

Choral Effect

- The background chorus ("ahhh…") was the band members singing individual notes, overlaid to create a choral effect

2000s

1970s

- In 1975 they didn't have computers to help them. All effects were made by physically splicing 16-track tape loops, taking *weeks*
- Click here for an interesting 10-minute doc about it from 2009

Last Week: Waves on a String

- Last time, with a bit of work, we derived the wave equation for waves on an open string

$$\frac{d^2y}{dt^2} = \frac{T}{\rho}\cdot\frac{d^2y}{dx^2} = v^2\cdot\frac{d^2y}{dx^2}, \qquad \text{where } v = \sqrt{\frac{T}{\rho}}$$

- Describes the motion of an oscillating string as a function of time *t* and position *x*. It has two solutions:

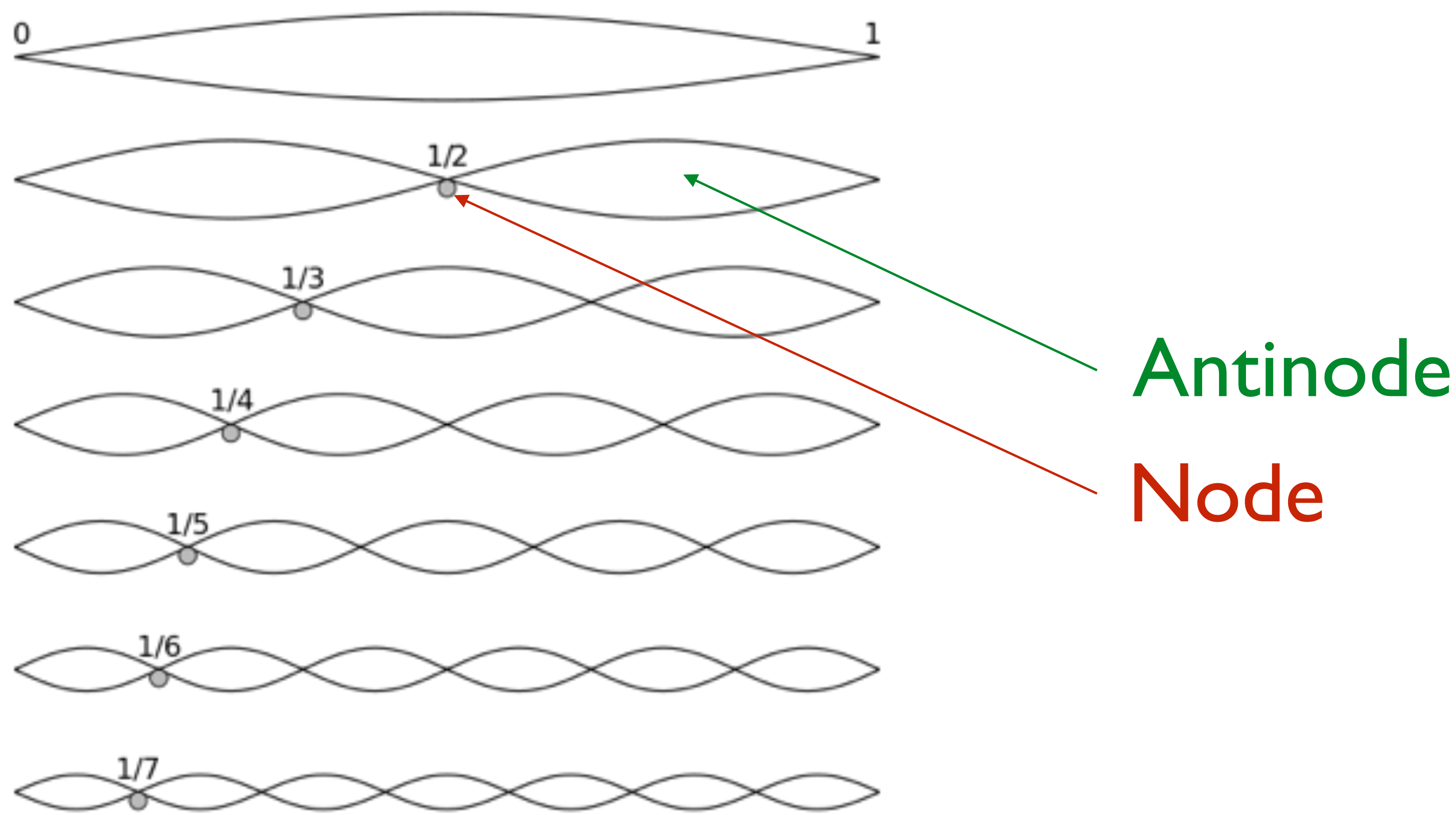
$$y(x,t) = A\sin(kx \pm \omega t)$$

$$= A\sin\frac{2\pi}{\lambda}(x \pm vt), \qquad \text{where } v = \lambda f = \sqrt{\frac{T}{\rho}}$$

- These are traveling waves moving to the right and to the left

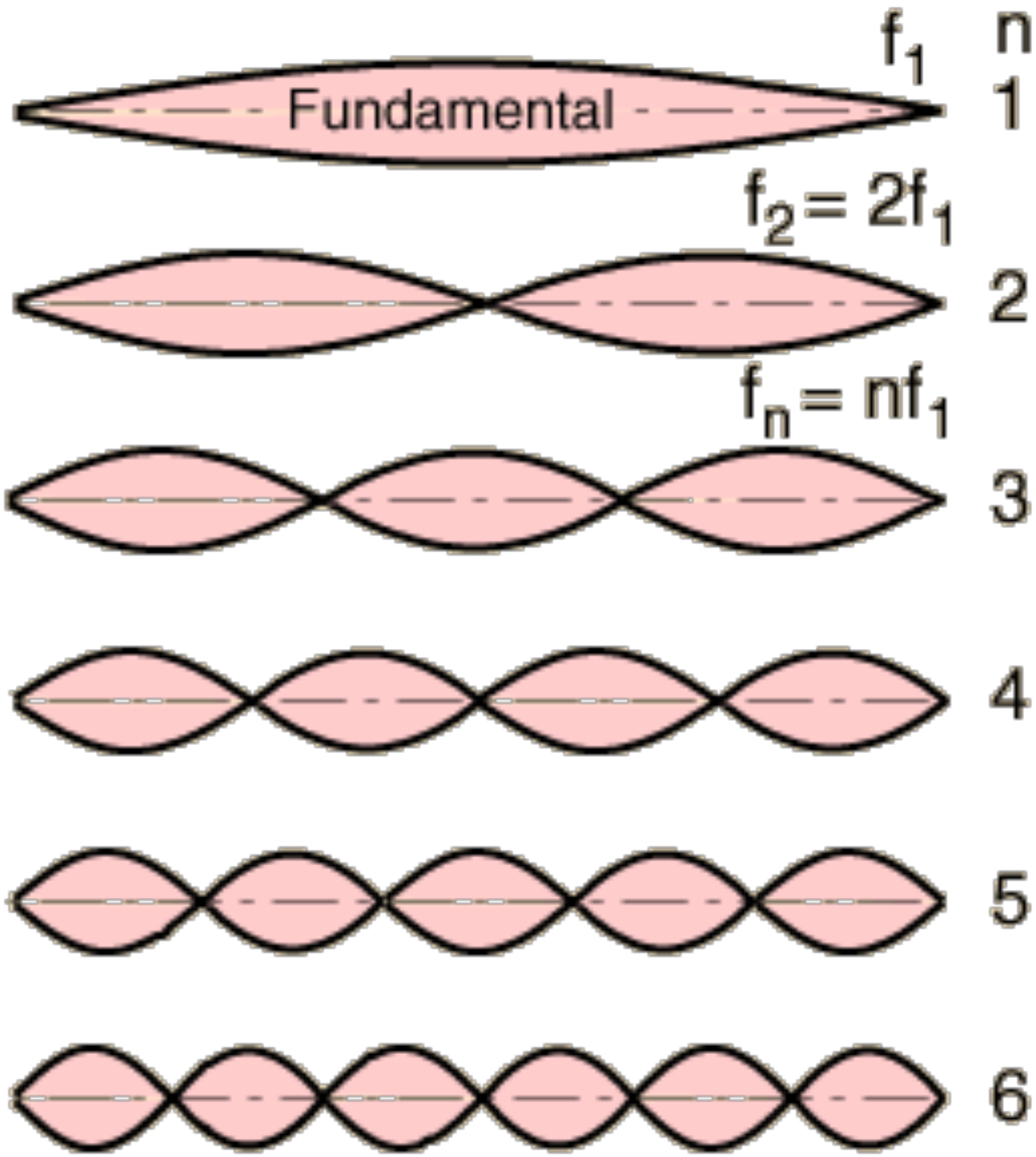
Standing Waves

- On a string with both ends fixed, you can set up standing waves by driving the string at the correct frequency



- The waves are the resonant superposition of traveling waves reflecting from the ends of the string with $v = \sqrt{T/\rho}$

Harmonics



$L = \lambda/2$ $\quad f_1 = v / \lambda = v/2L$

$L = \lambda$ $\quad f_2 = v/L = 2f_1$

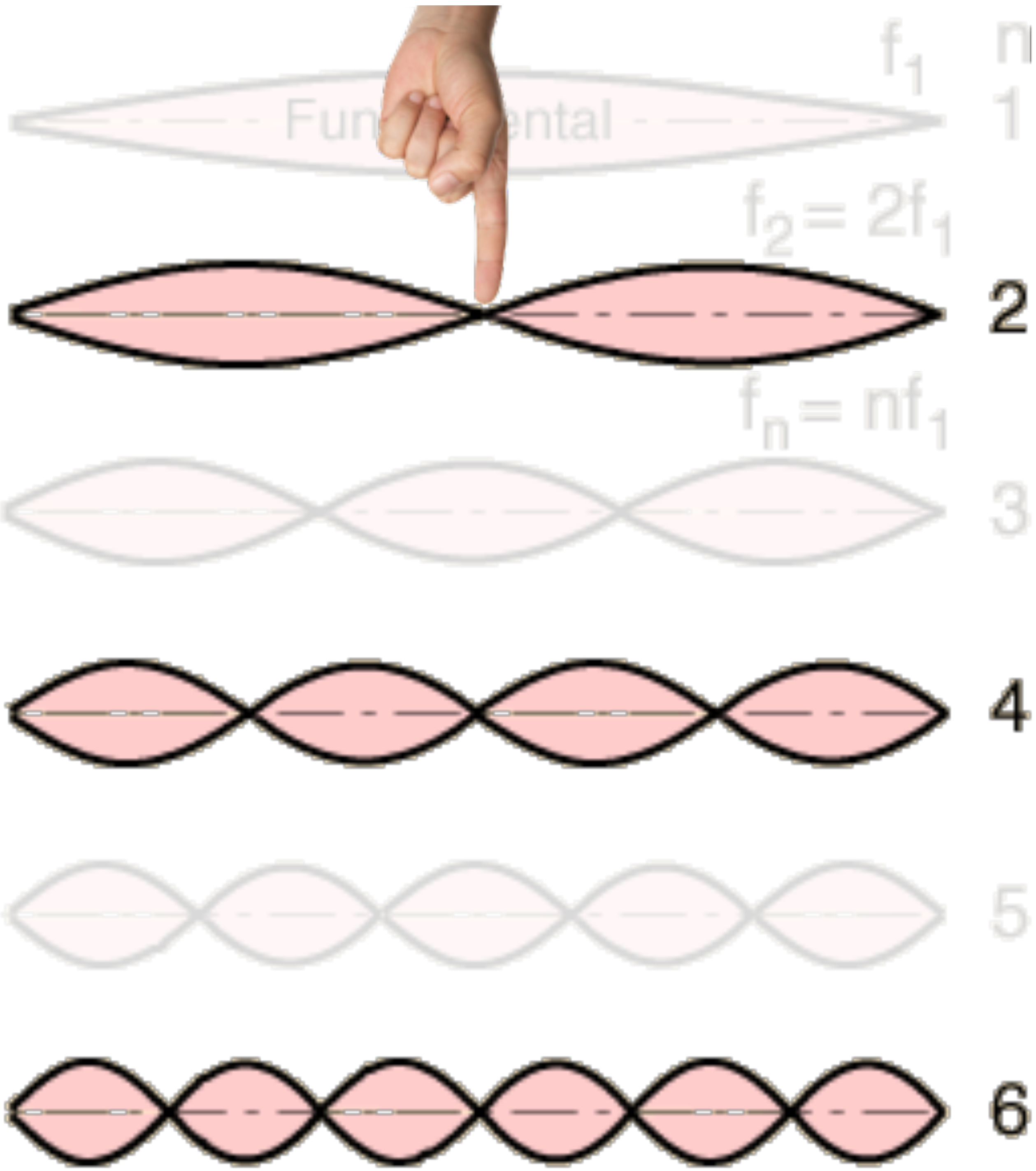
$L = 3\lambda/2$ $\quad f_3 = 3v/2L = 3f_1$

$L = 2\lambda$ $\quad f_4 = 2v/L = 4f_1$

$L = 5\lambda/2$ $\quad f_5 = 5v/2L = 5f_1$

$L = 3\lambda$ $\quad f_6 = 3v/L = 6f_1$

Harmonics



- You can cause the string to vibrate differently to change the timbre
- If a string is touched at its midpoint, it can only vibrate at frequencies with a node at the midpoint
- The odd-integer harmonics (including the fundamental frequency) are suppressed

Music Terminology

- Instrumental tones are made up of sine waves
- Harmonic: an integer multiple of the fundamental frequency of the tone
- Partial: any one of the sine waves making up a complex tone. Can be harmonic, but doesn't have to be
- Overtone: any partial in the tone except for the fundamental. Again, doesn't have to be harmonic
- Inharmonicity: deviation of any partial from an ideal harmonic. Many acoustic instruments have inharmonic partials. Do you know which ones?

Fourier Analysis

- Fourier's Theorem: any reasonably continuous periodic function can be decomposed into a sum of sinusoids (sine and cosine functions):

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$$

$$= a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + a_n \cos n\omega t + \ldots$$

$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots + b_n \sin n\omega t + \ldots$$

- The sum can be (but doesn't have to be) infinite
- The series is called a Fourier series

Fourier Coefficients

- The coefficients a_n and b_n determine the shape of the final waveform. Musically, they determine the harmonic partials contributing to a sound
- Mathematical definition of the coefficients:

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\,dt$$

avg. of *f*(*t*) × cosine

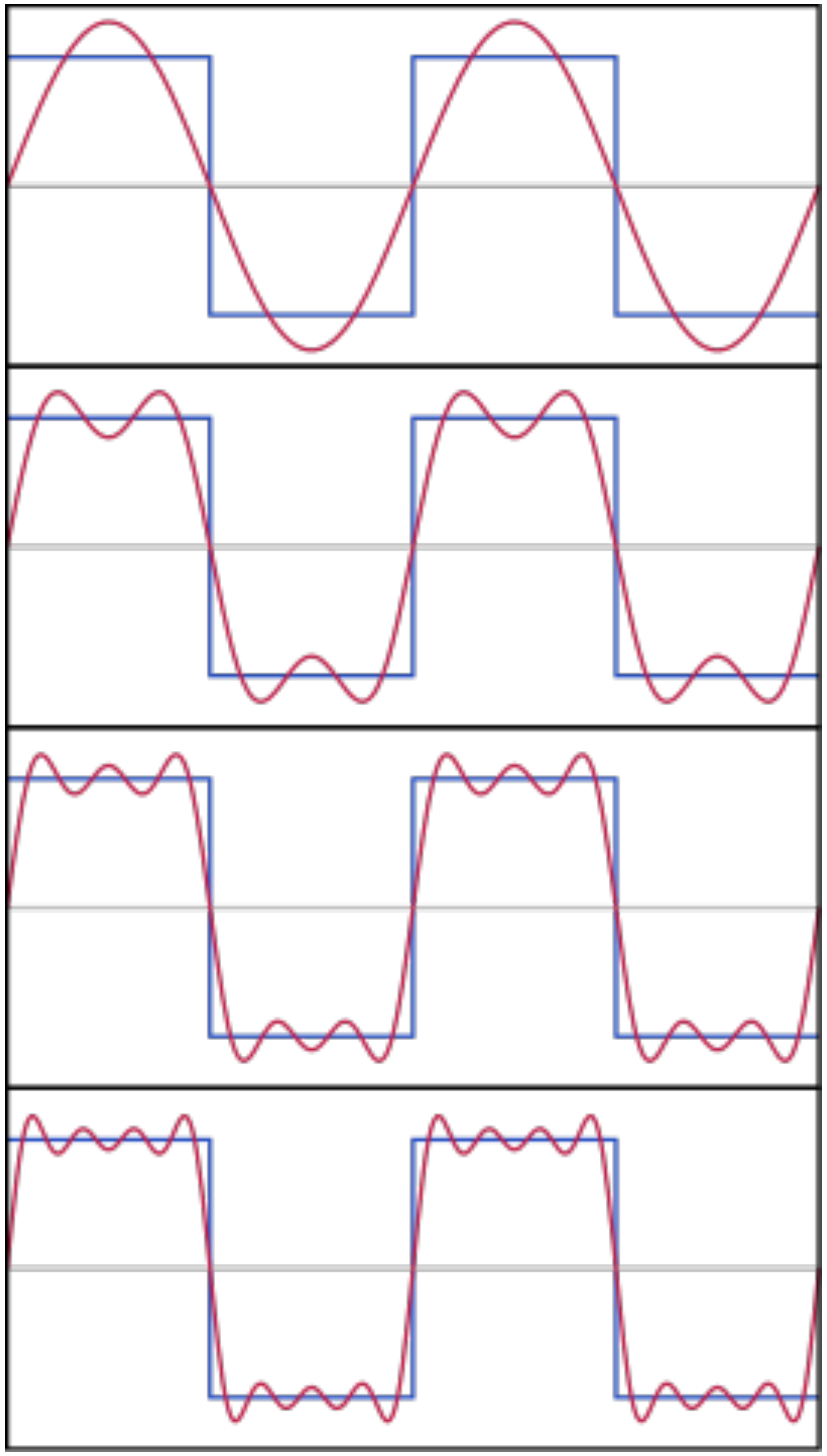
$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)\,dt$$

avg. of *f*(*t*) × sine

$$\omega = 2\pi / \tau$$

Visualization: Square Wave

- A square wave oscillates between two constant values
- E.g., voltage in a digital circuit
- Fourier's Theorem: the square pulse can be built up from a set of sinusoidal functions
- Not every term contributes equally to the sum
- I.e., the a_k and b_k differ to produce the final waveform



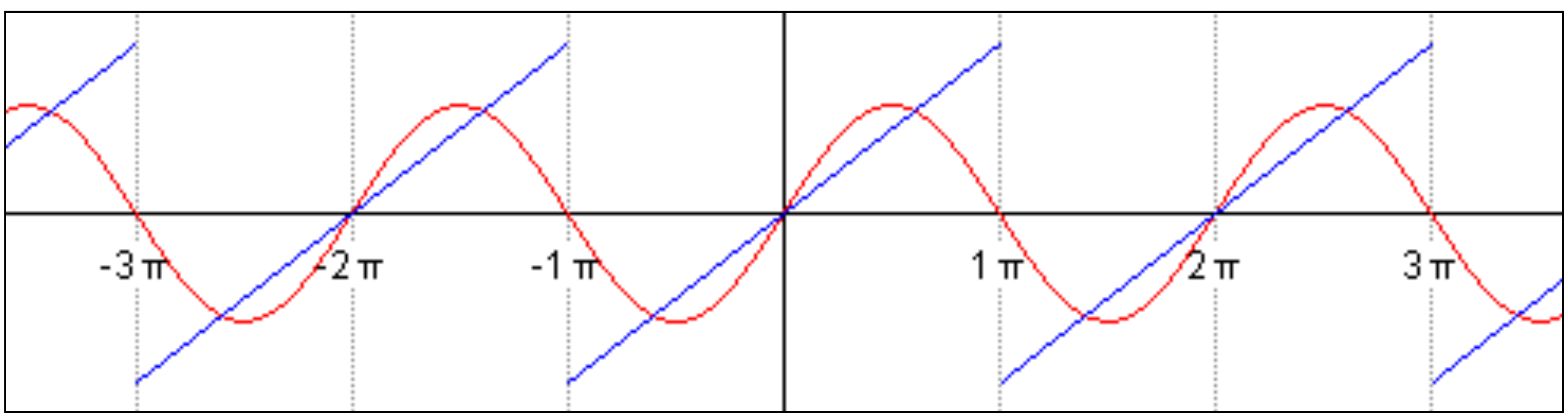
Visualization: Sawtooth Wave

- The sawtooth waveform represents the function

$$f(t) = t/\pi, \quad -\pi \le t < \pi$$

$$f(t+2\pi n) = f(t), \quad -\infty < t < \infty, \ n = 0,1,2,3,\ldots$$

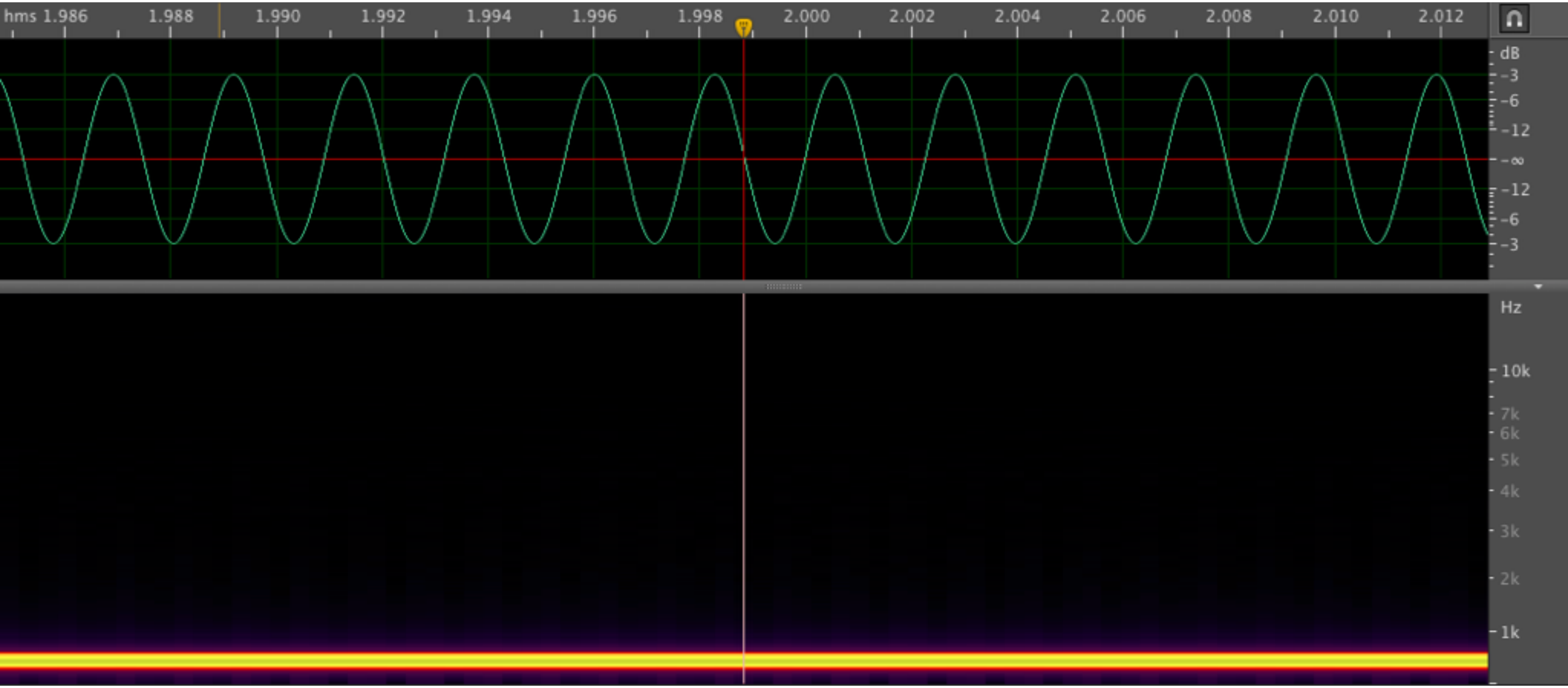
- Also called a "ramp" function, used in synthesizers. Adding more terms gives a better approximation



440 Hz Sine Wave

- The 440 Hz sine wave (A4 on the piano) is a pure tone

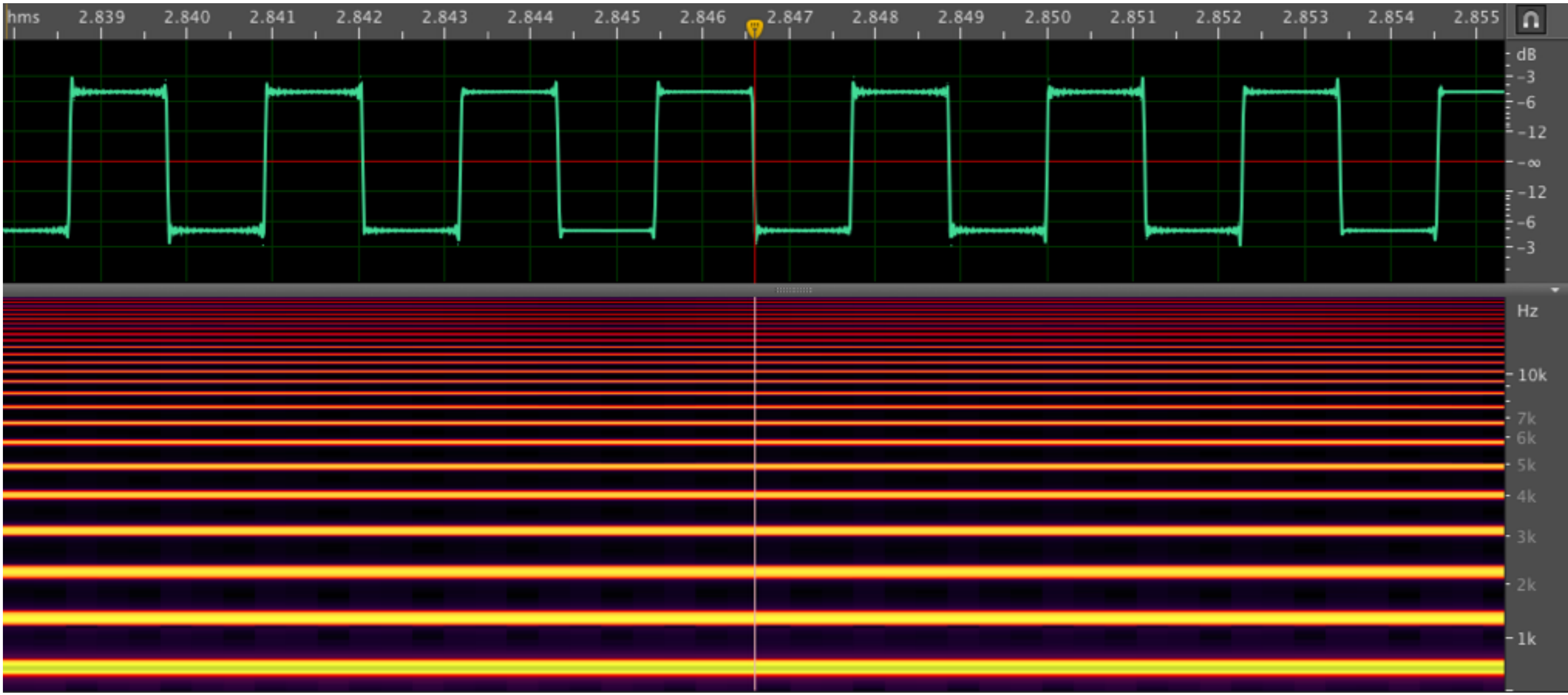
http://www.audiocheck.net/audiofrequencysignalgenerator_index.php



440 Hz Square Wave

- The square wave is built from the fundamental plus a truncated series of the higher harmonics

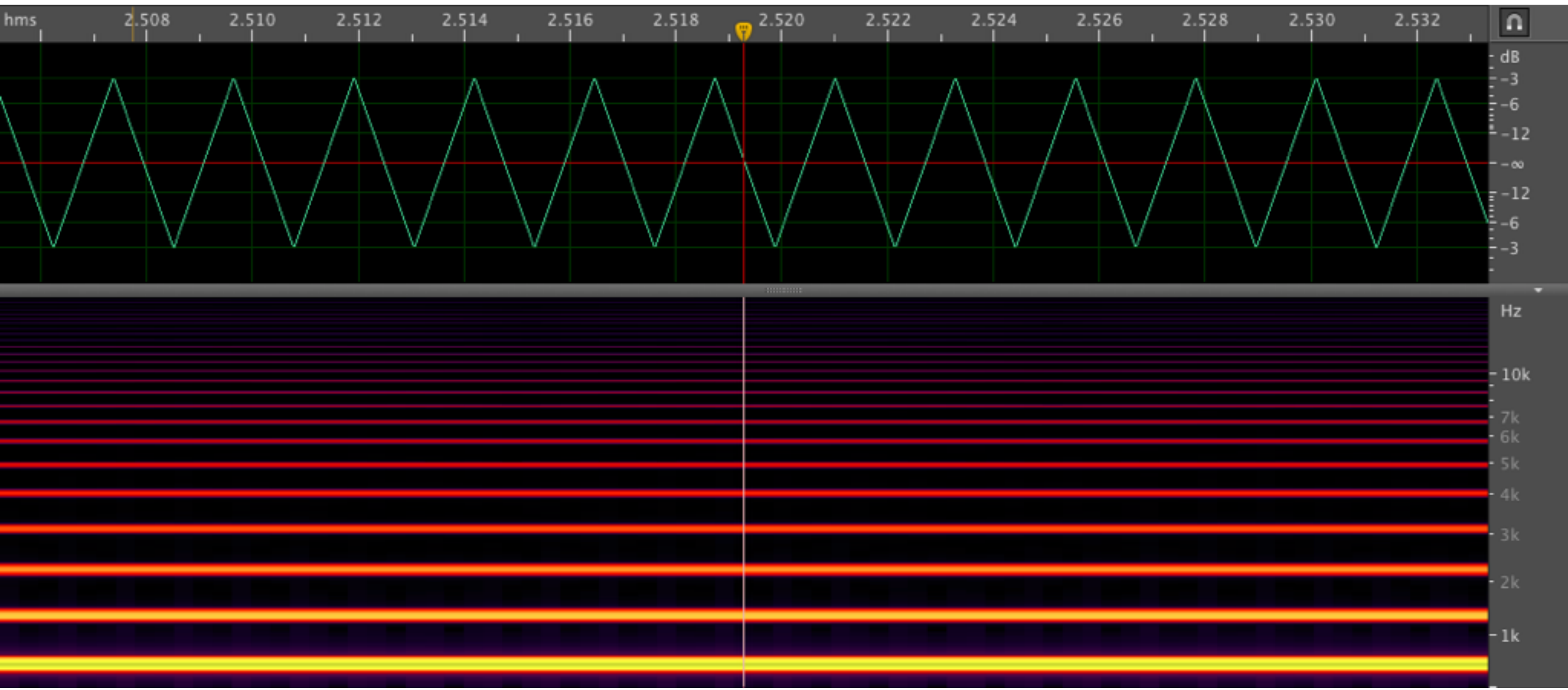
http://www.audiocheck.net/audiofrequencysignalgenerator_index.php



440 Hz Triangle Wave

- The triangle wave is also built from a series of the higher harmonics

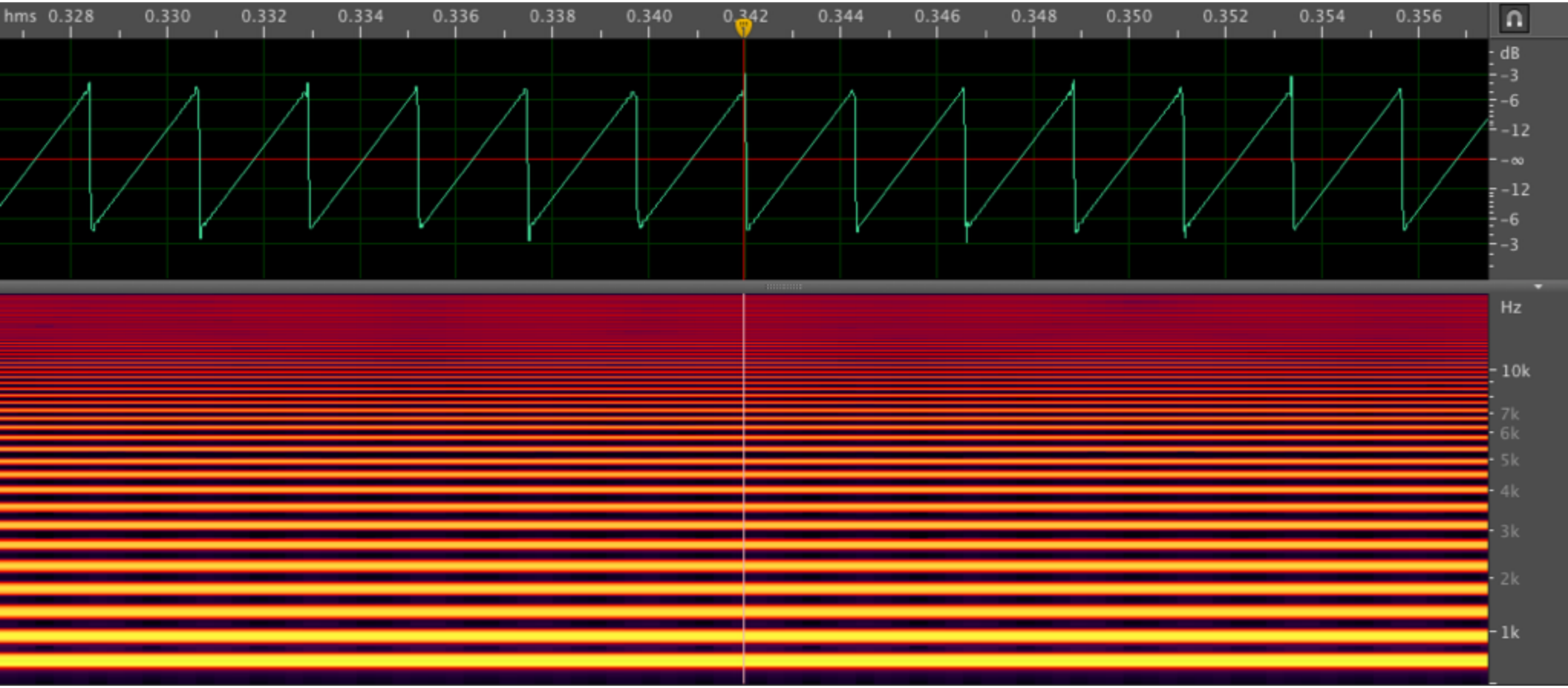
http://www.audiocheck.net/audiofrequencysignalgenerator_index.php



440 Hz Sawtooth

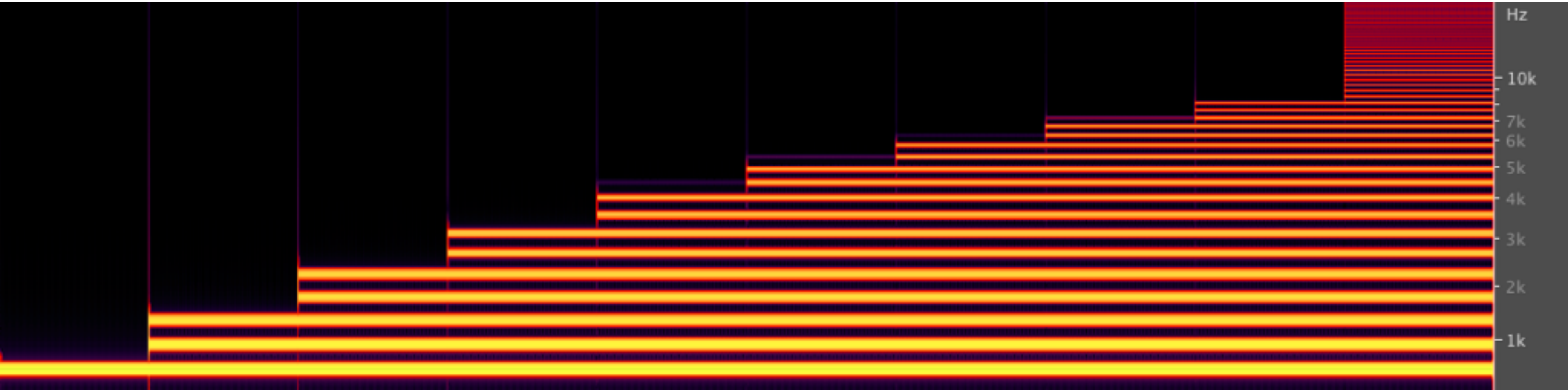
- The sawtooth waveform: not a particularly pleasant sound…

http://www.audiocheck.net/audiofrequencysignalgenerator_index.php



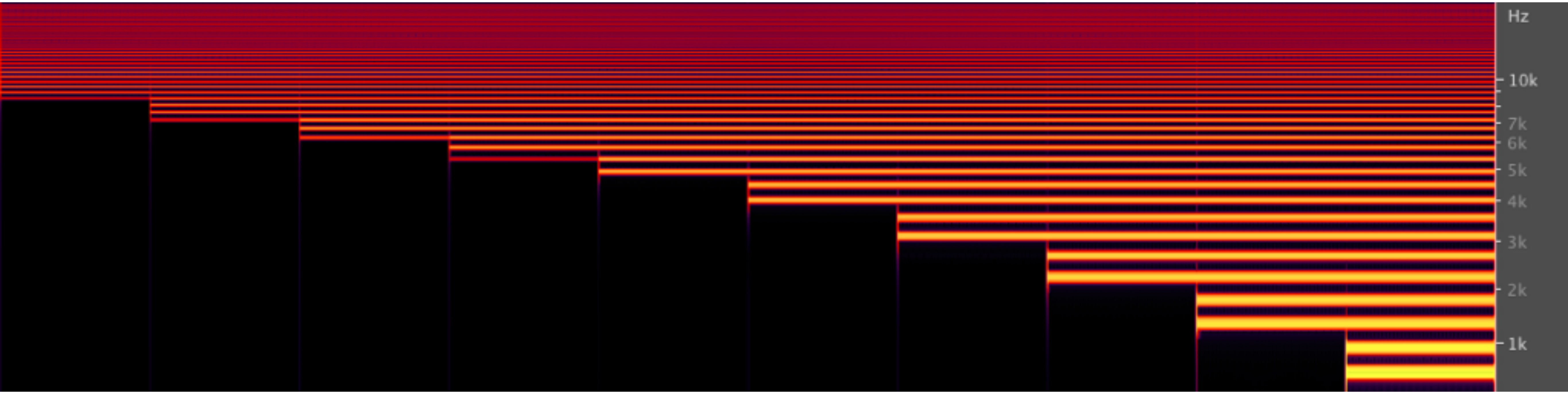
Building Up a Sawtooth

- In this 10 s clip we will hear a sawtooth waveform being built up from its harmonic partials
- Notice how the higher terms make the sawtooth sound increasingly shrill (or "bright")



Building Up a Sawtooth

- In the second clip we hear the sawtooth being built up from its highest frequencies first
- The sound of the sawtooth is clearly dominated by the fundamental frequency



Partials in Different Waveforms

- You observed different waveforms produced by a function generator

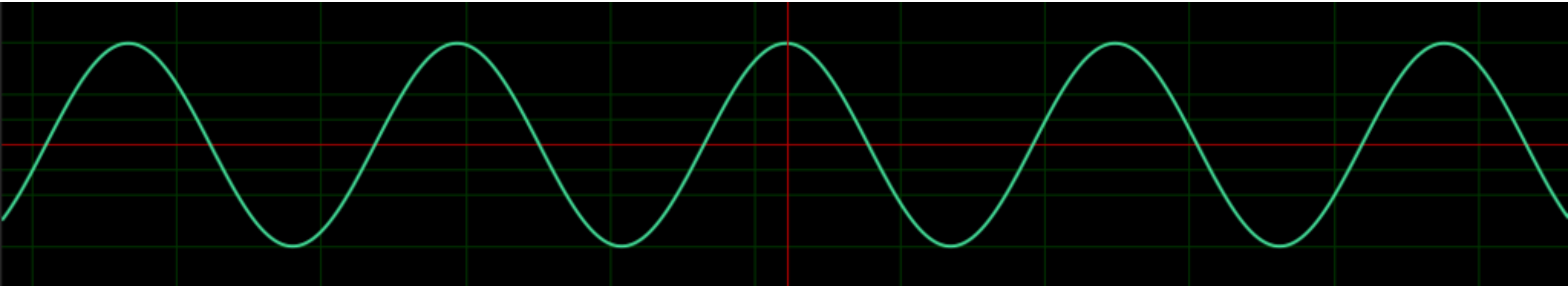
- In the generator the square and triangle waves are produced by adding Fourier components
- See this document for a description of how it's actually done

Contributing Partials

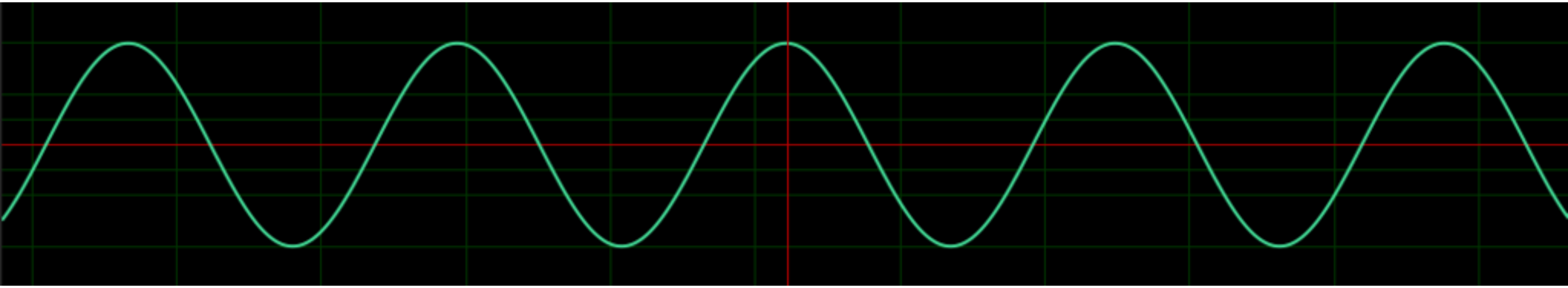
- Question: are all harmonic partials present in every waveform?
- Without performing the Fourier decomposition, how can we tell?
- Shortcut: use the reflection symmetry of the waveform $f(t)$ about the point $t = 0$
- Why? Because of the underlying reflection symmetry of the partials that make up a wave

Even Functions: $f(x) = f(-x)$

- Cosines are symmetric about their midpoint:

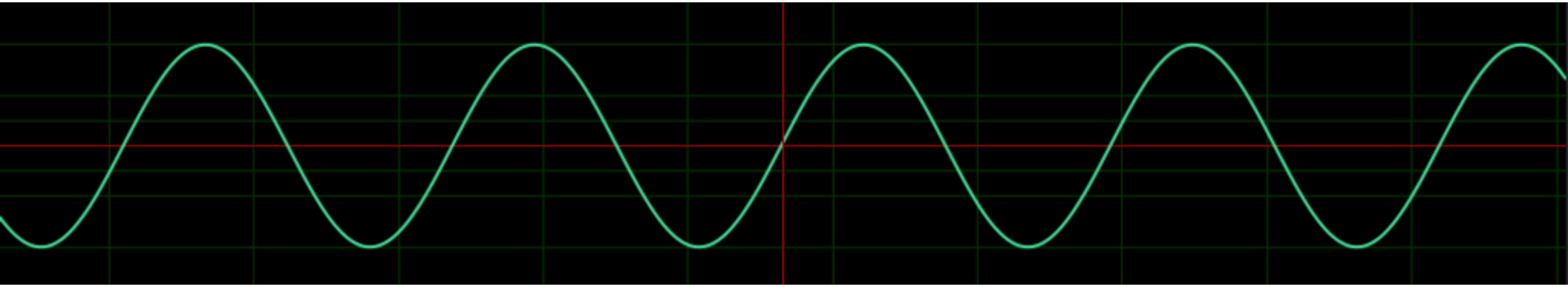


- Reflecting about the midpoint maps the cosine onto itself

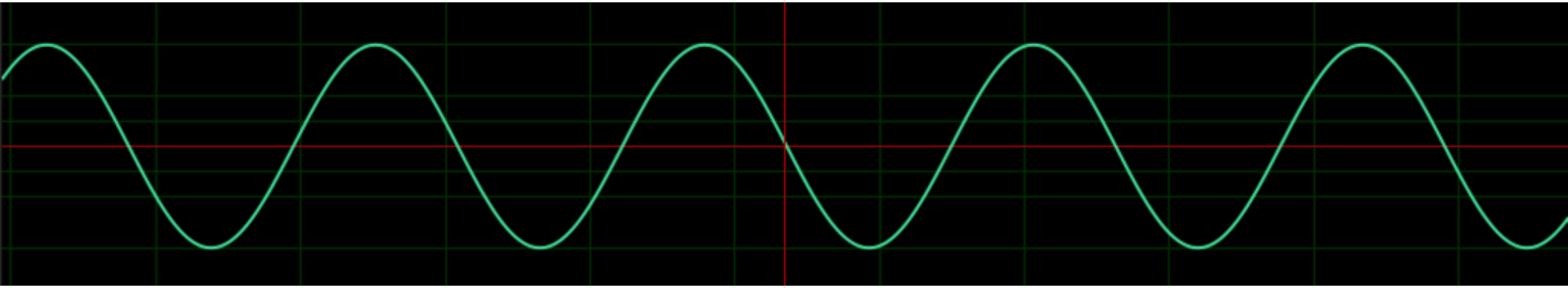


Odd Functions: $f(-x) = -f(x)$

- Sines are anti-symmetric about their midpoint:



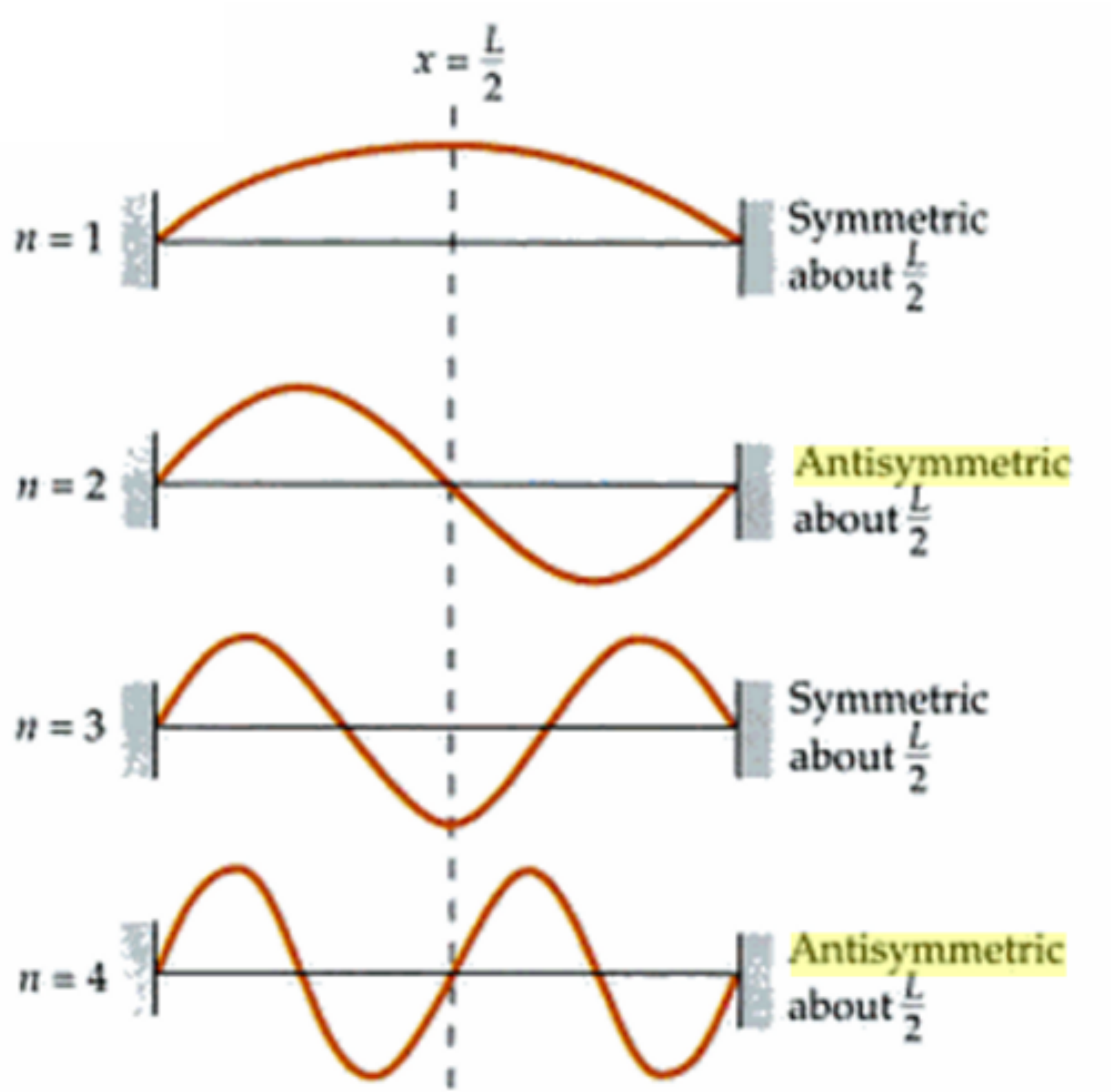
- Reflecting about the midpoint flips the sin upside down



Exploiting Symmetry

- Combining even and odd functions is like combining numbers:
  - Even x Even = Even
  - Odd x Odd = Even
  - Odd x Even = Odd
- So if we have a waveform $f(t)$ that is odd or even we can predict the contributing partials because we know that
  - a_n ~ average of $f(t)$ x cosine
  - b_n ~ average of $f(t)$ x sine

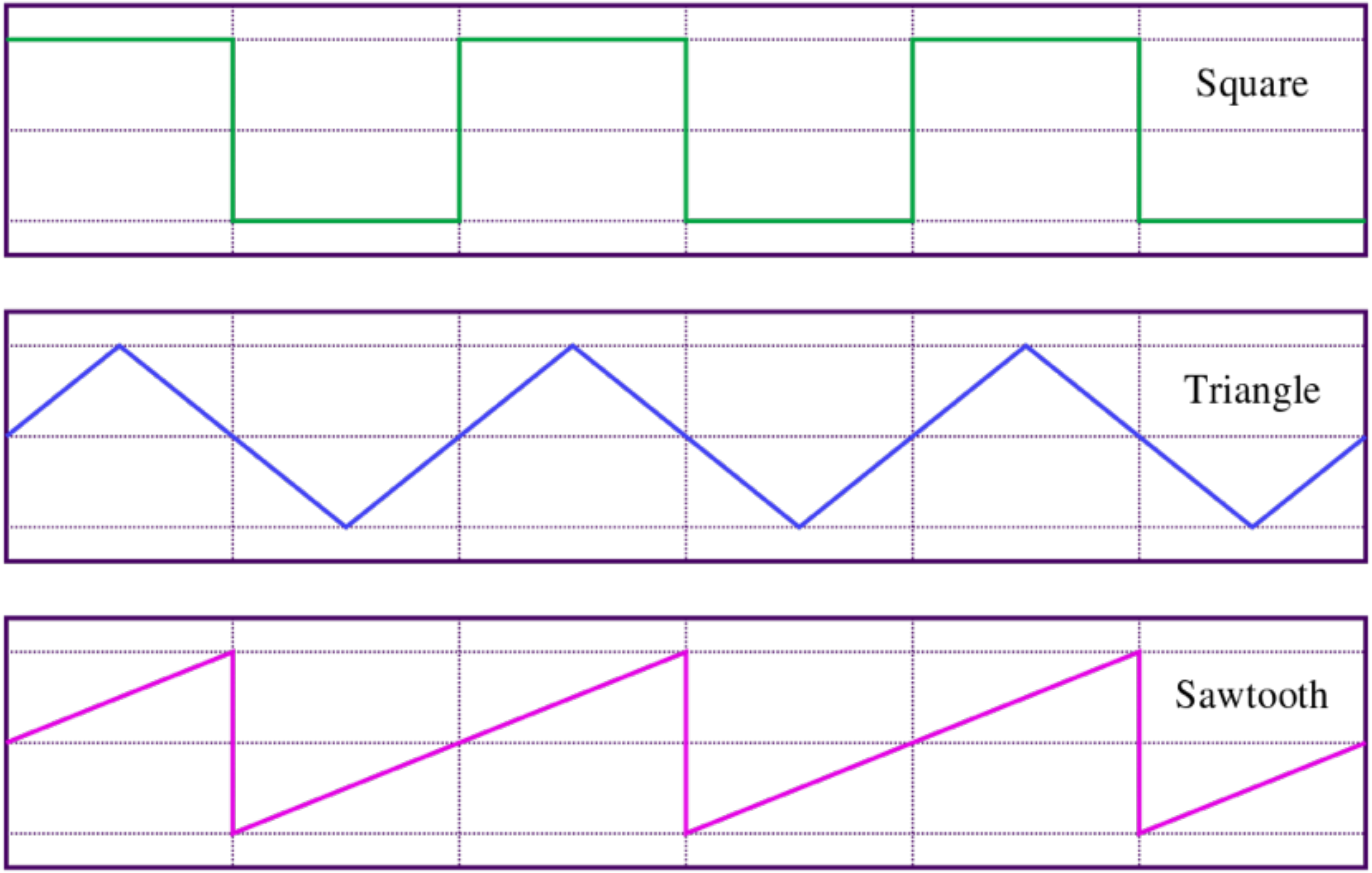
Odd/Even Harmonics



From Tipler and Mosca

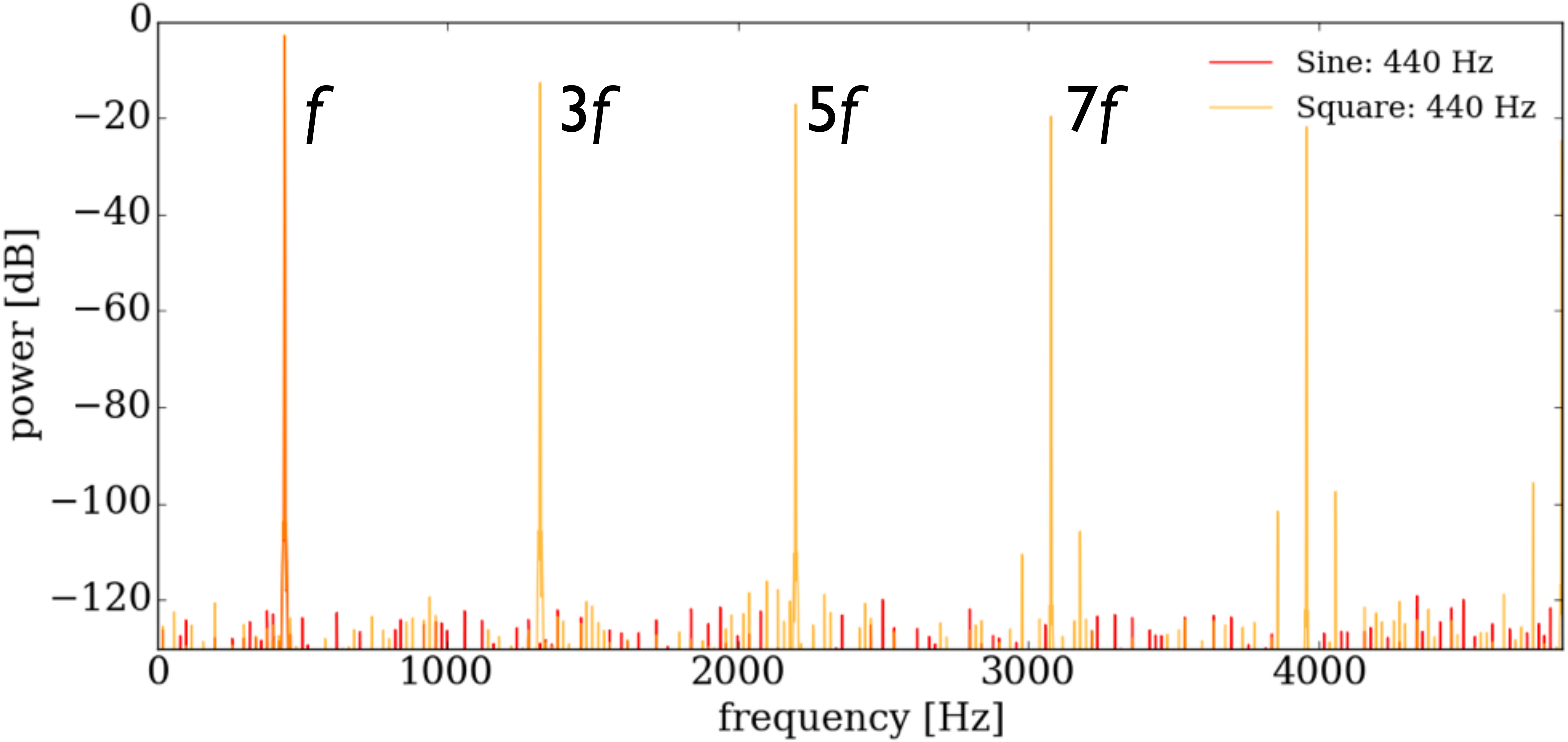
- In a plucked string, the odd harmonics are symmetrical about the center (even)
- The even harmonics are anti-symmetrical (odd)
- Symmetric (even) waveforms only contain odd harmonics
- Anti-symmetric (odd) waveforms **must** contain even harmonics, but can also include odd ones

Which Partials Contribute?



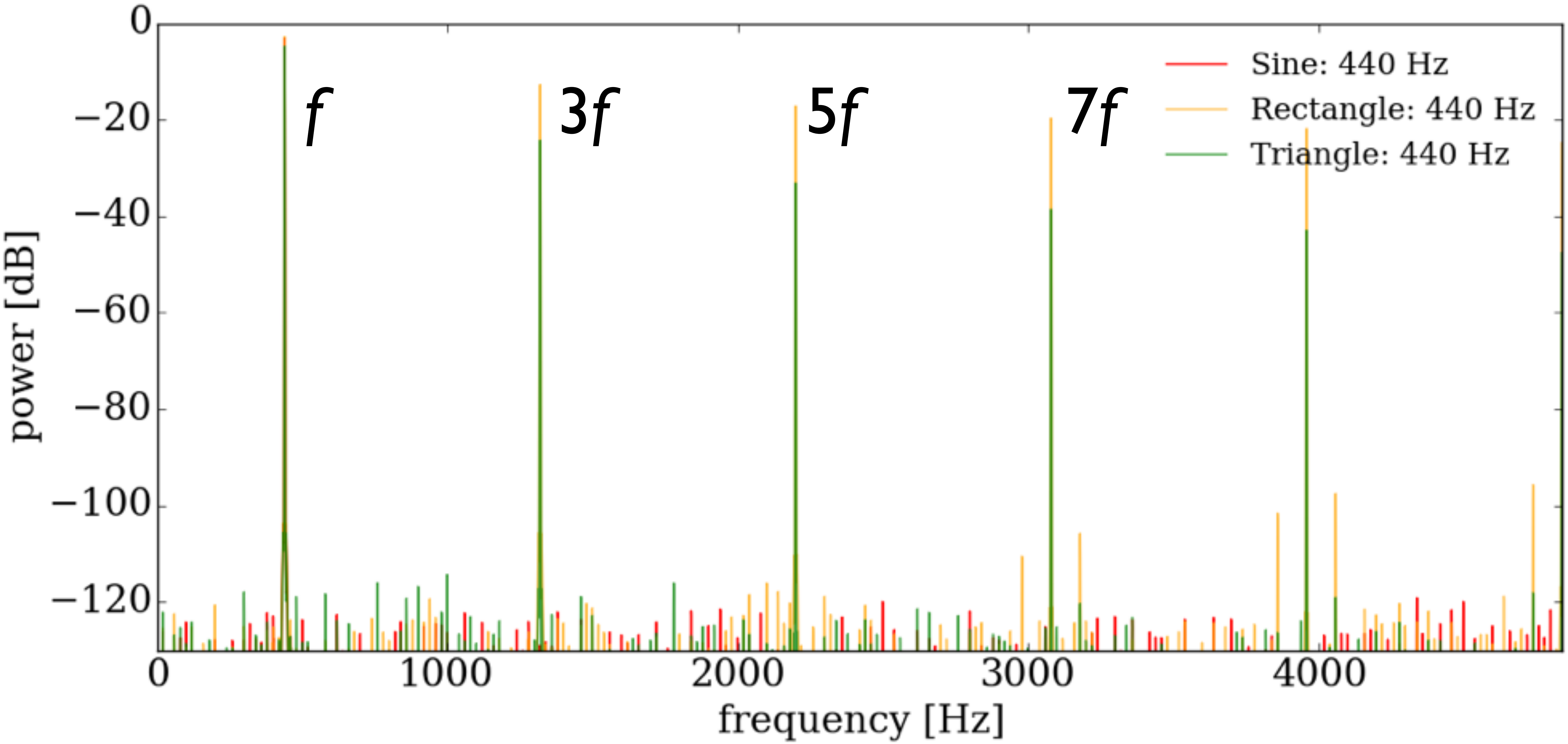
Square Wave

- Which harmonics are present in the square wave?



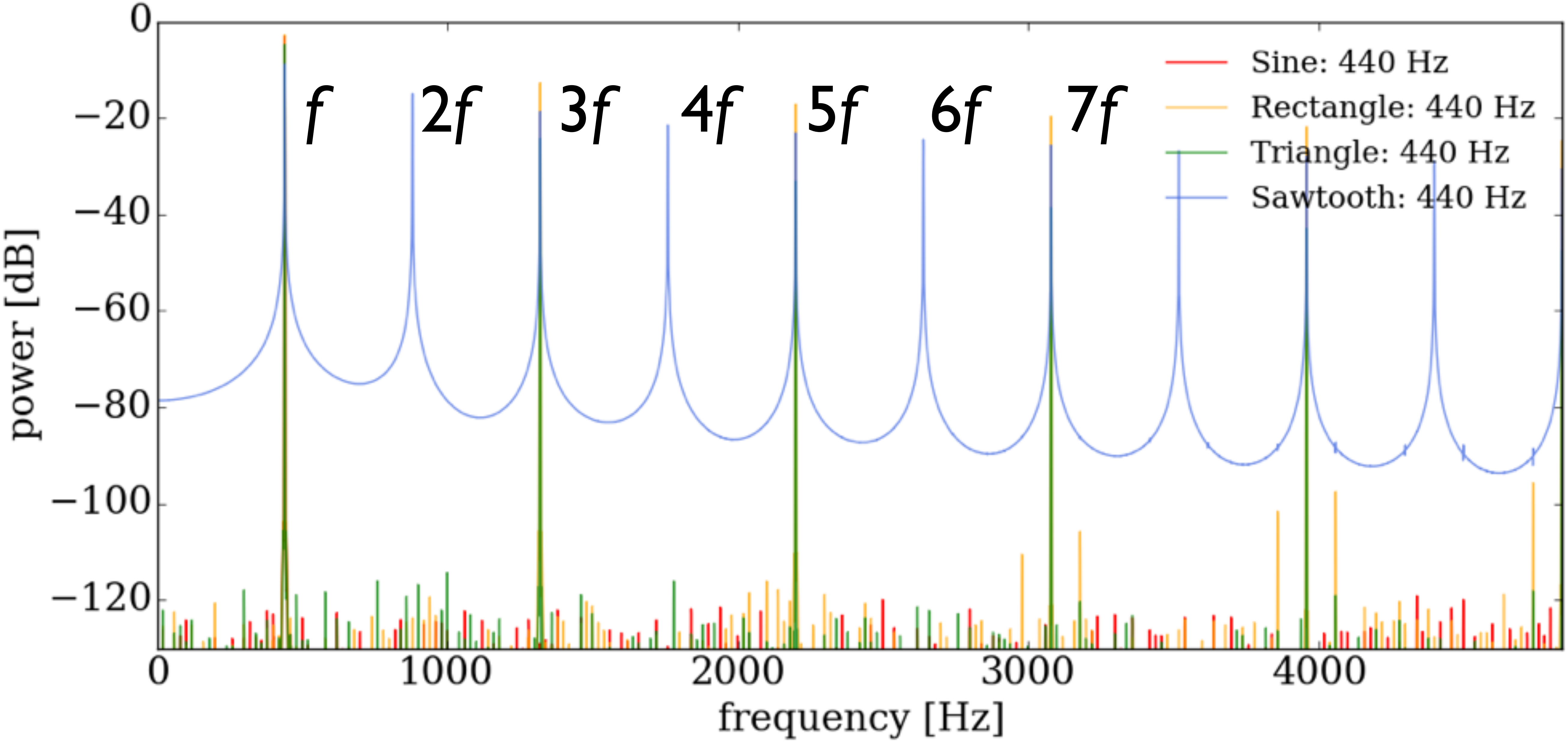
Triangle Wave

- Which harmonics are present in the triangle wave?



Sawtooth Wave

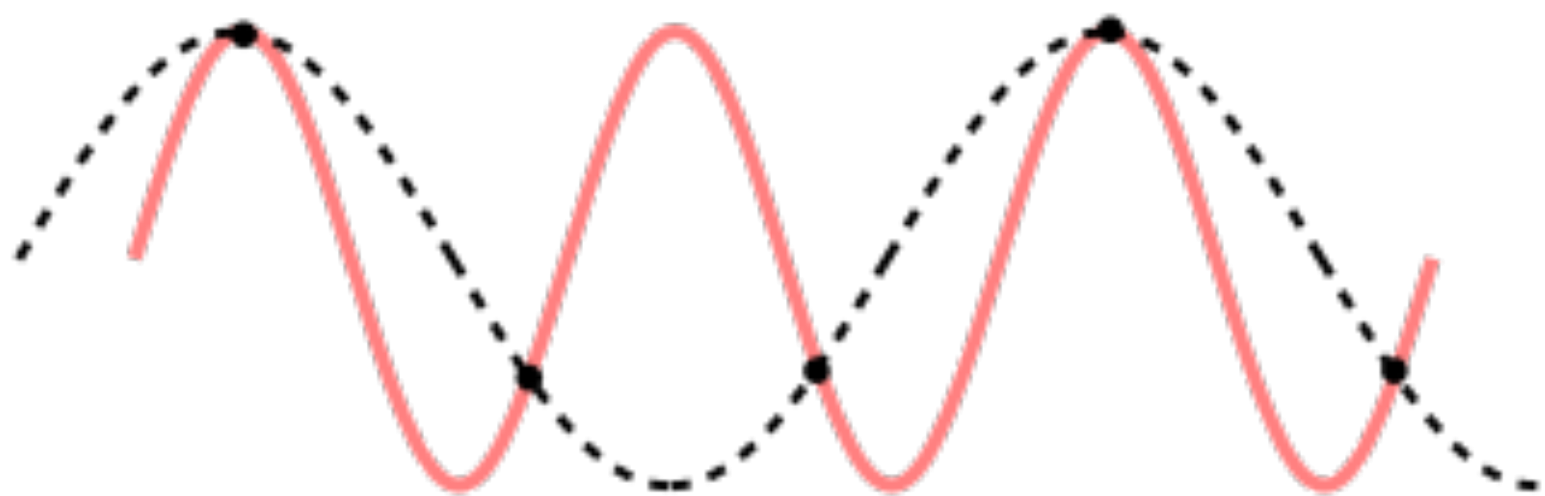
- Which harmonics are present in the sawtooth wave?



Waveform Sampling

Sampling and Digitization

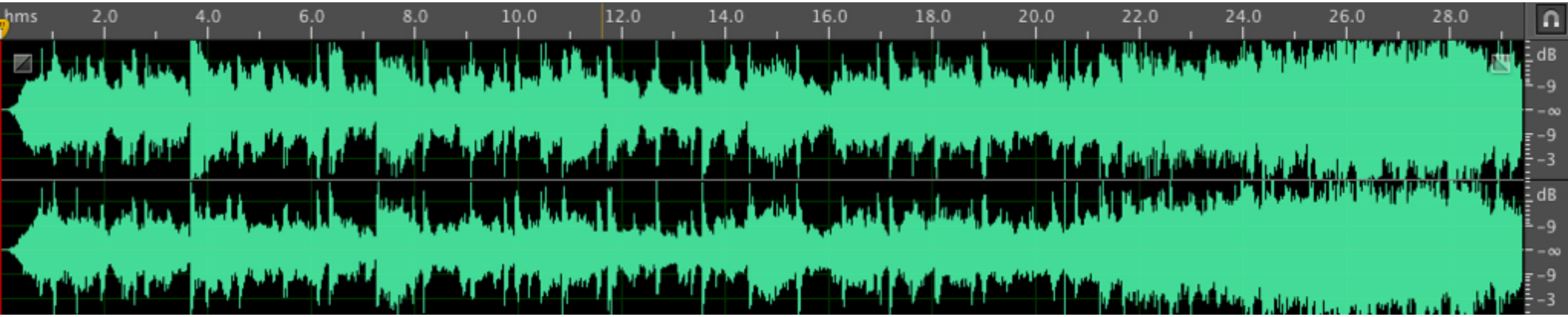
- When we digitize a waveform we have to take care to make sure the sampling rate is sufficiently high



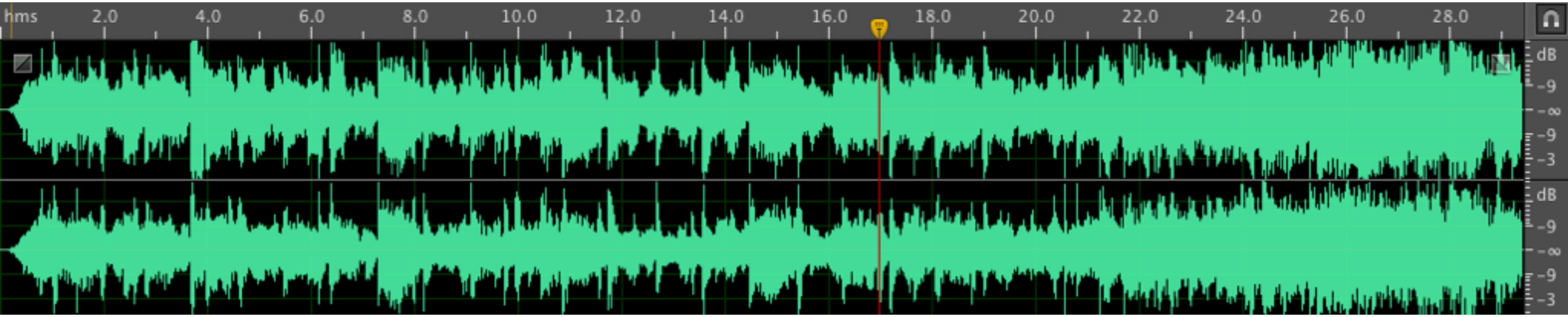
- If we don't use sufficient sampling, high-frequency and lower-frequency components can be confused
- This is a phenomenon called aliasing

Sampling Rate and Fidelity

- Song from start of the class with 44 kHz sampling

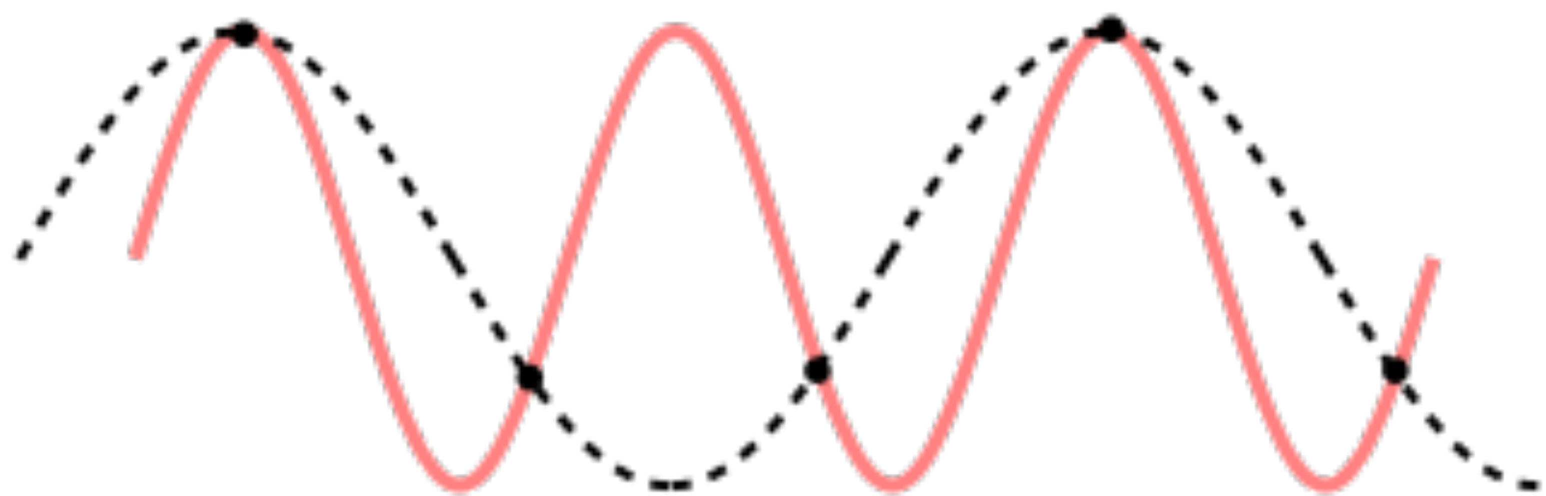


- Same song, now with 6 kHz sampling rate. What is the difference (if any)?



Nyquist Limit

- If you sample a waveform with frequency f_S, you are guaranteed a perfect reconstruction of all components up to $f_S/2$



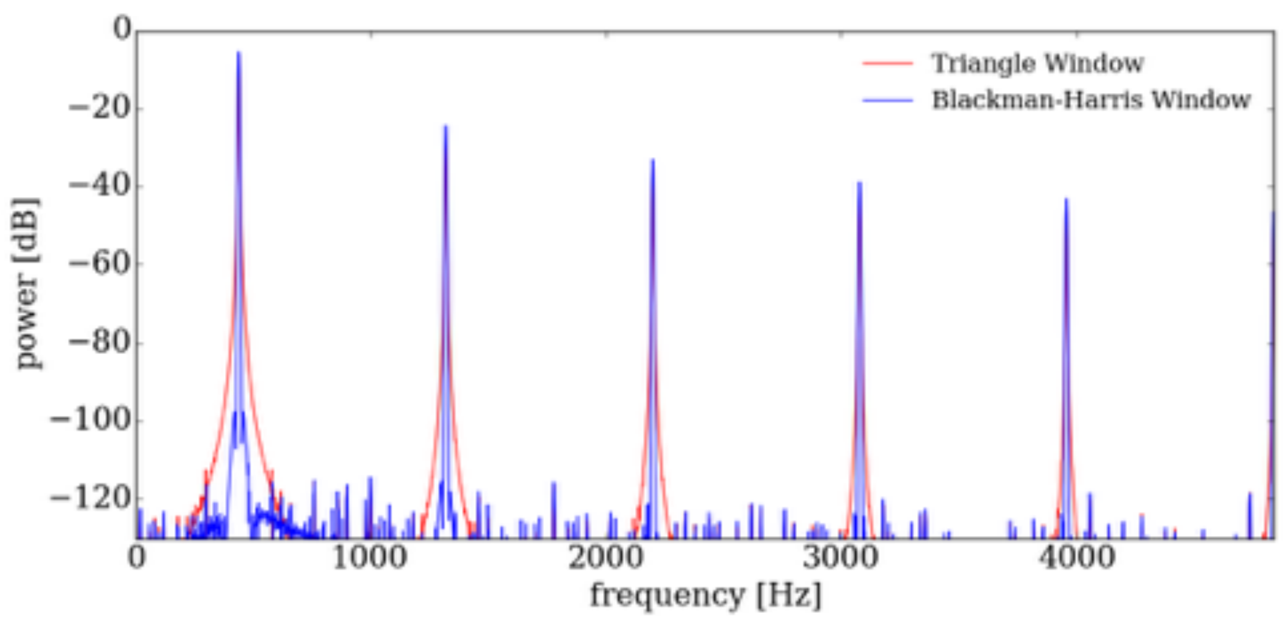
- So with 44 kHz sampling, we reconstruct signals up to 22 kHz
- With 6 kHz sampling, we alias signals >3 kHz
- What is the typical frequency range of human hearing? Does this explain the difference in what you heard?

Fast Fourier Transform (FFT)

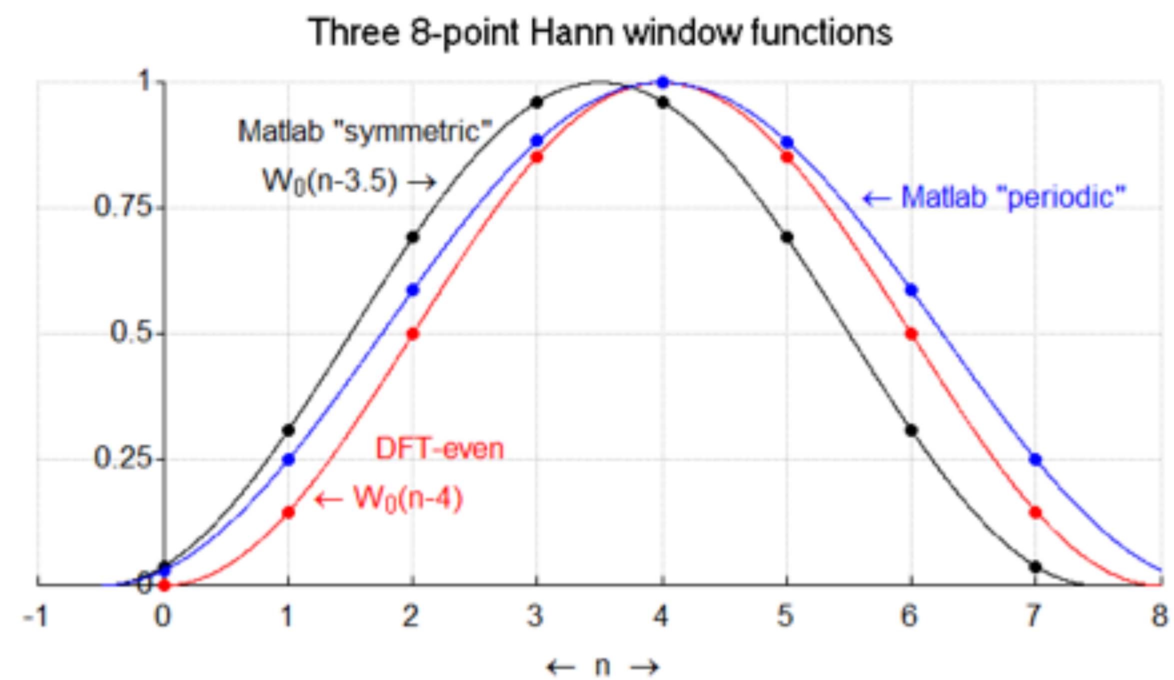
- The Adobe Audition program (and it's freeware version Audacity) will perform a Fourier decomposition for you
- On the computer we can't represent continuous functions; everything is discrete
- The Fourier decomposition is accomplished using an algorithm called the Fast Fourier Transform (FFT)
  - Works really well if you have N data points, where N is some power of 2: $N = 2^k$, $k = 0, 1, 2, 3, \ldots$
  - If N is not a power of two, the algorithm will pad the end of the data set with zeros

Calculating the FFT

- When you calculate an FFT, you have freedom to play with a couple of parameters:
  - The number of points in your data sample, *N*
  - The window function used

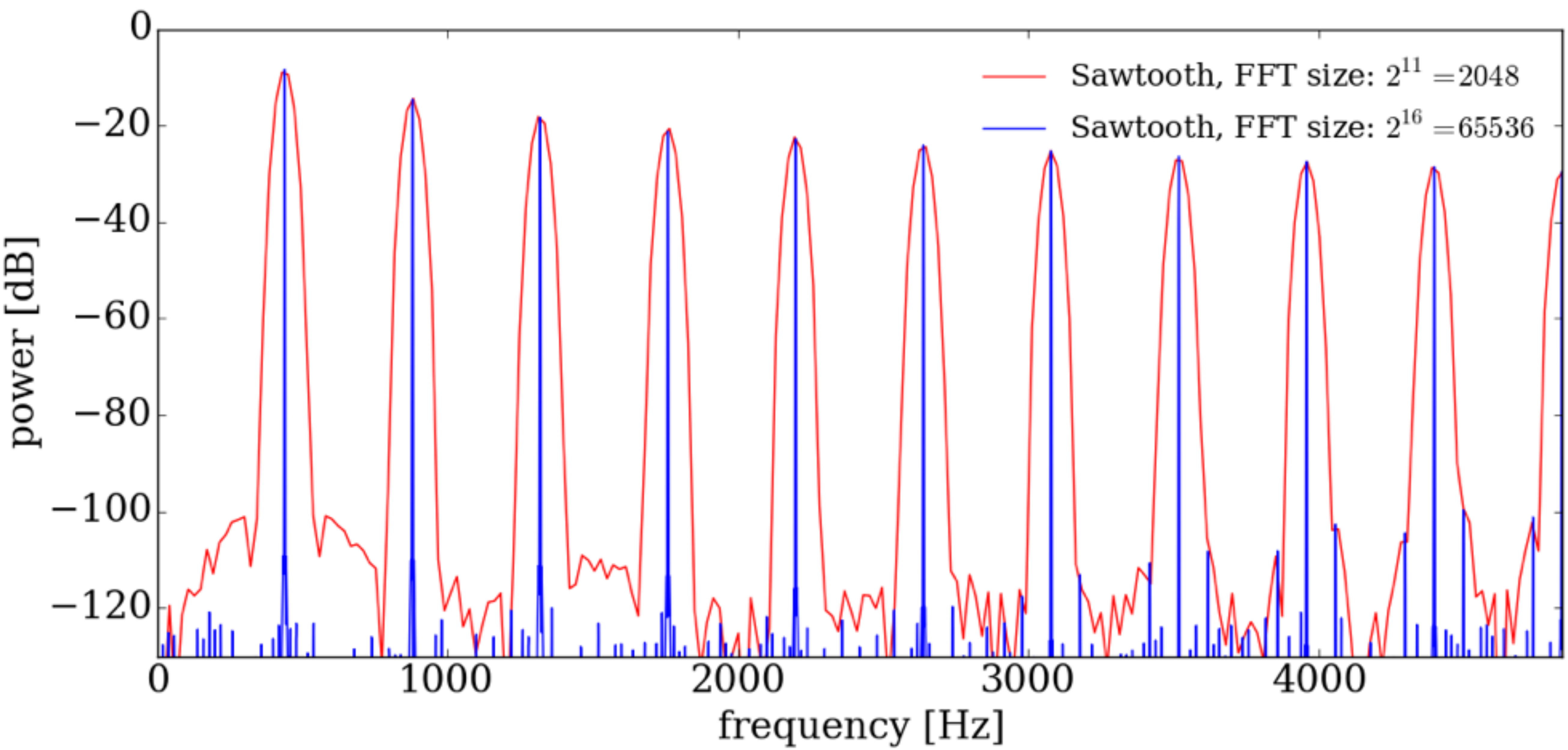


×



Effect of FFT Size

- Larger *N* = better resolution of harmonic peaks



Uncertainty Principle

- Why does a longer data set produce a better resolution in the frequency domain?
- Time-Frequency Uncertainty Principle:

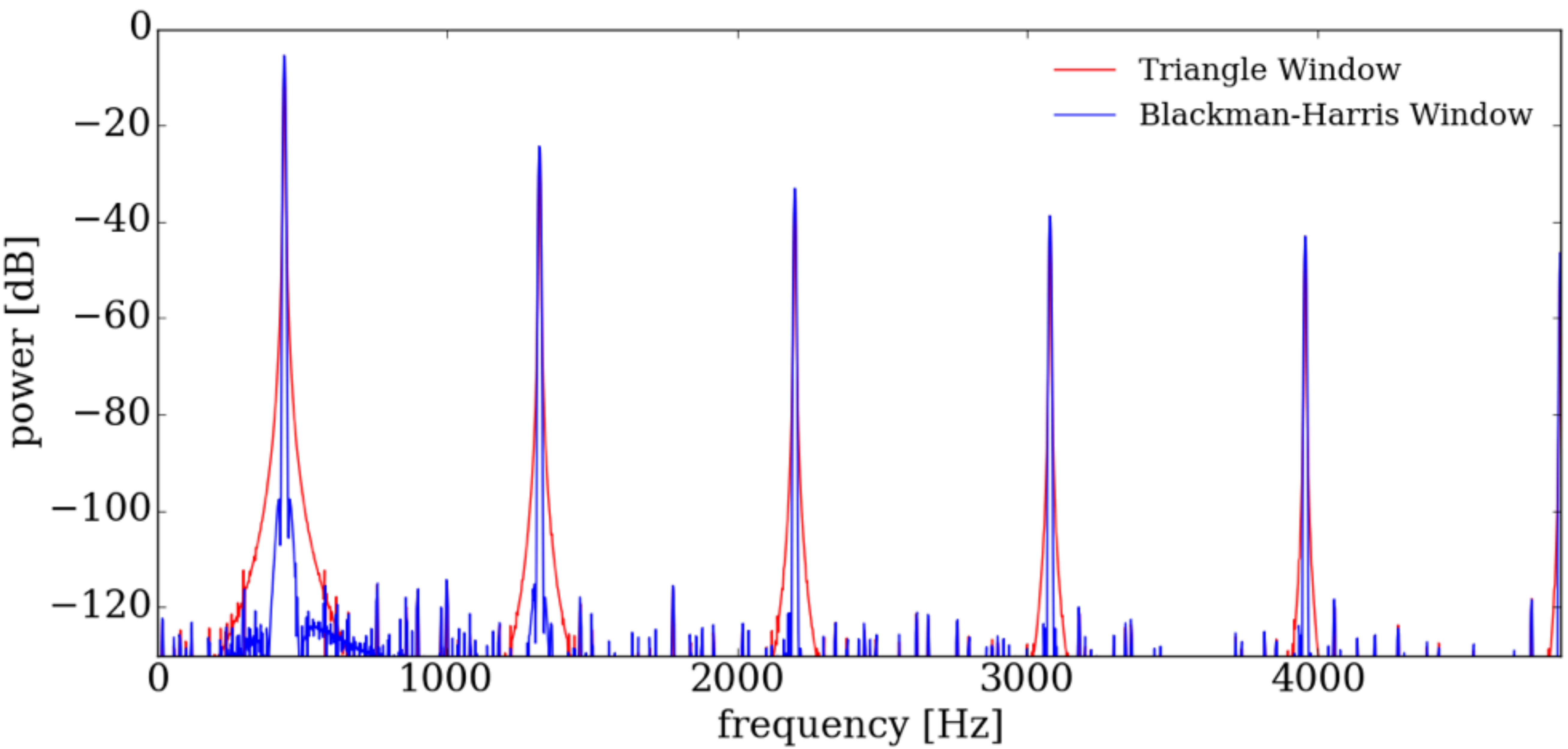
$$\Delta t \cdot \Delta f \sim 1$$

Localization of measurement in *time* — Localization of measurement in *frequency*

- Localizing the waveform in time (small *N*, and therefore small **Δ***t*) leads to a big uncertainty in frequency (**Δ***f*)
- Localizing the frequency (small **Δ***f*) leads means less localization of the waveform in time (large **Δ***t*)

Effect of Window Function

- Certain windows can give you better frequency resolution



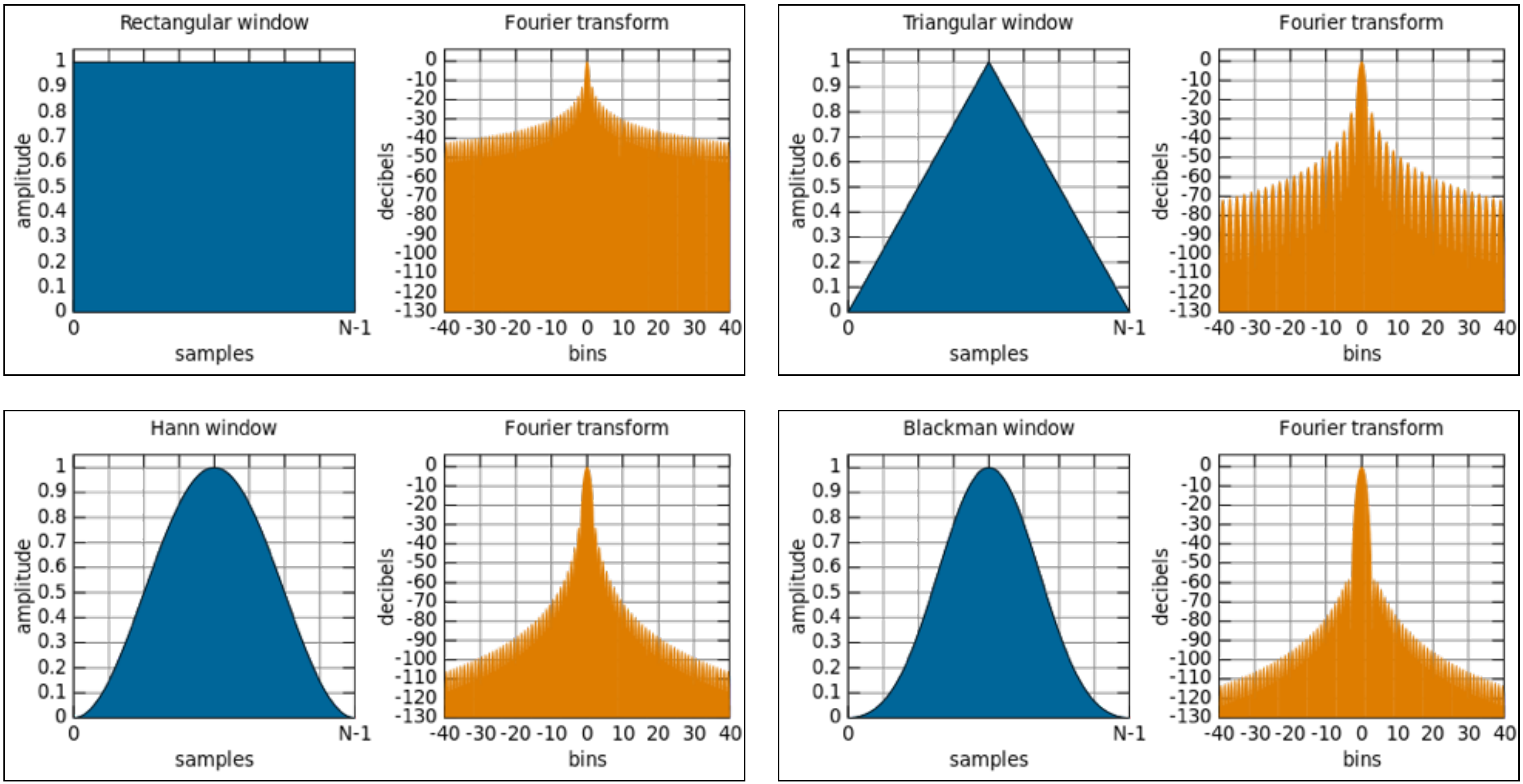
Windowing

- Why do we use a window function at all?
  - Because the Fourier Transform is technically defined for periodic functions, which are defined out to $t = \pm\infty$
  - We don't have infinitely long time samples, but truncated versions of periodic functions
  - As a result, the FFT contains artifacts (sidebands) because we've "chopped off" the ends of the function
  - The window function mitigates the sidebands by going smoothly to zero in the time domain
  - Thus, our function doesn't drop sharply to zero at the start and end of the sample, giving a nicer FFT

Window Examples

- Time and frequency behavior of common windows:

Olli Niemitalo, commons.mediawiki.org



Summary

- The partials present in a complex tone contribute to the timbre of the sound
  - Partials can be harmonic (integer multiples of the fundamental frequency) or inharmonic
  - The high-frequency components affect the brightness of a sound
  - Use the reflection symmetry of the waveform $f(t)$ about t=0 to predict the partials which contribute to it
- Fourier's Theorem:
  - Any reasonably continuous periodic function can be expressed in terms of a sum of sinusoidal functions (Fourier series)
  - The spectrograms we have been looking at are a discrete calculation of the Fourier components of signals (FFT)
  - You can play with the window function and size N of your FFT to improve the frequency resolution in your spectrograms