PHY 103: Standing Waves and Harmonics

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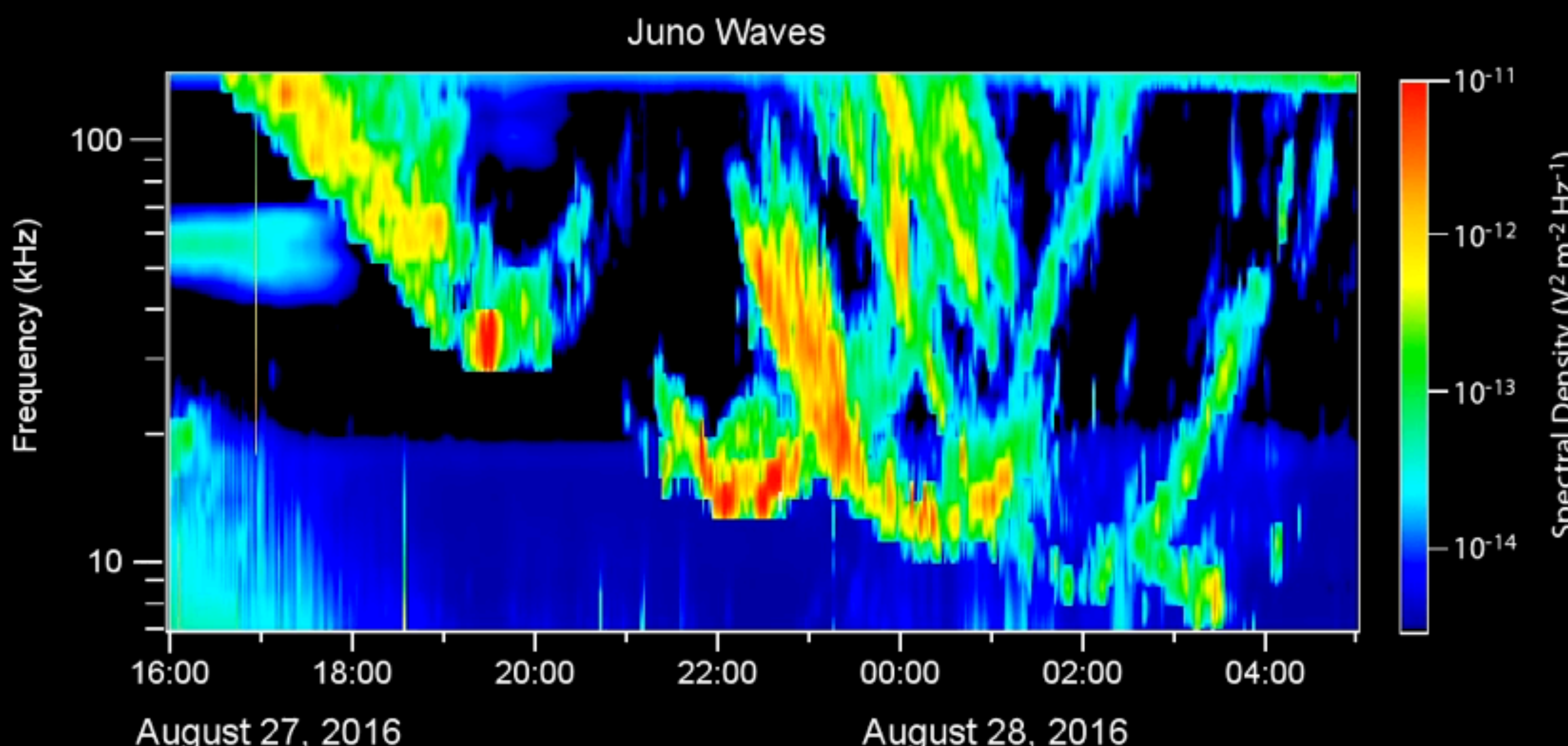
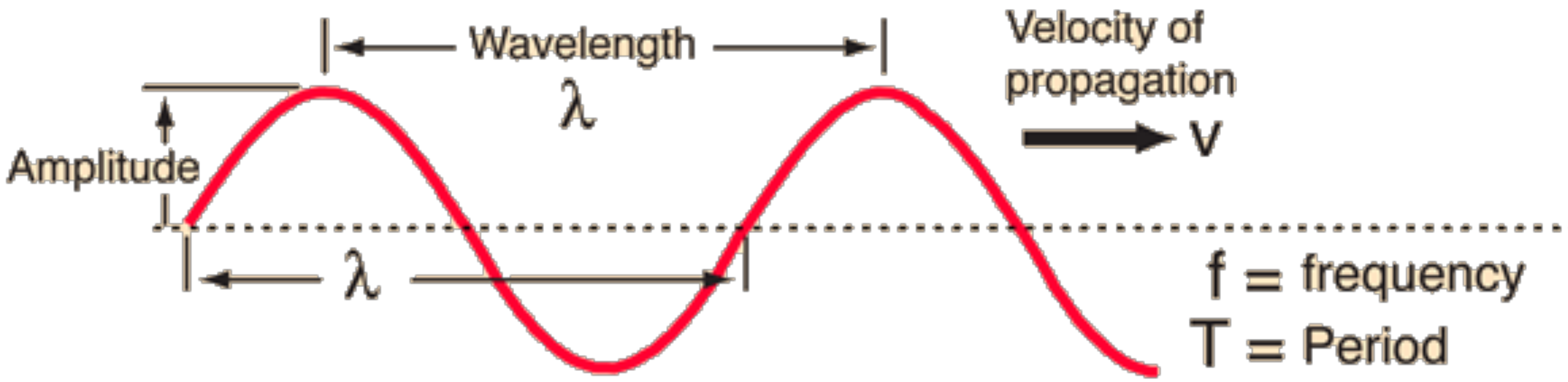
Sounds of the Universe…



NASA/JPL, September 2016

Properties of Waves



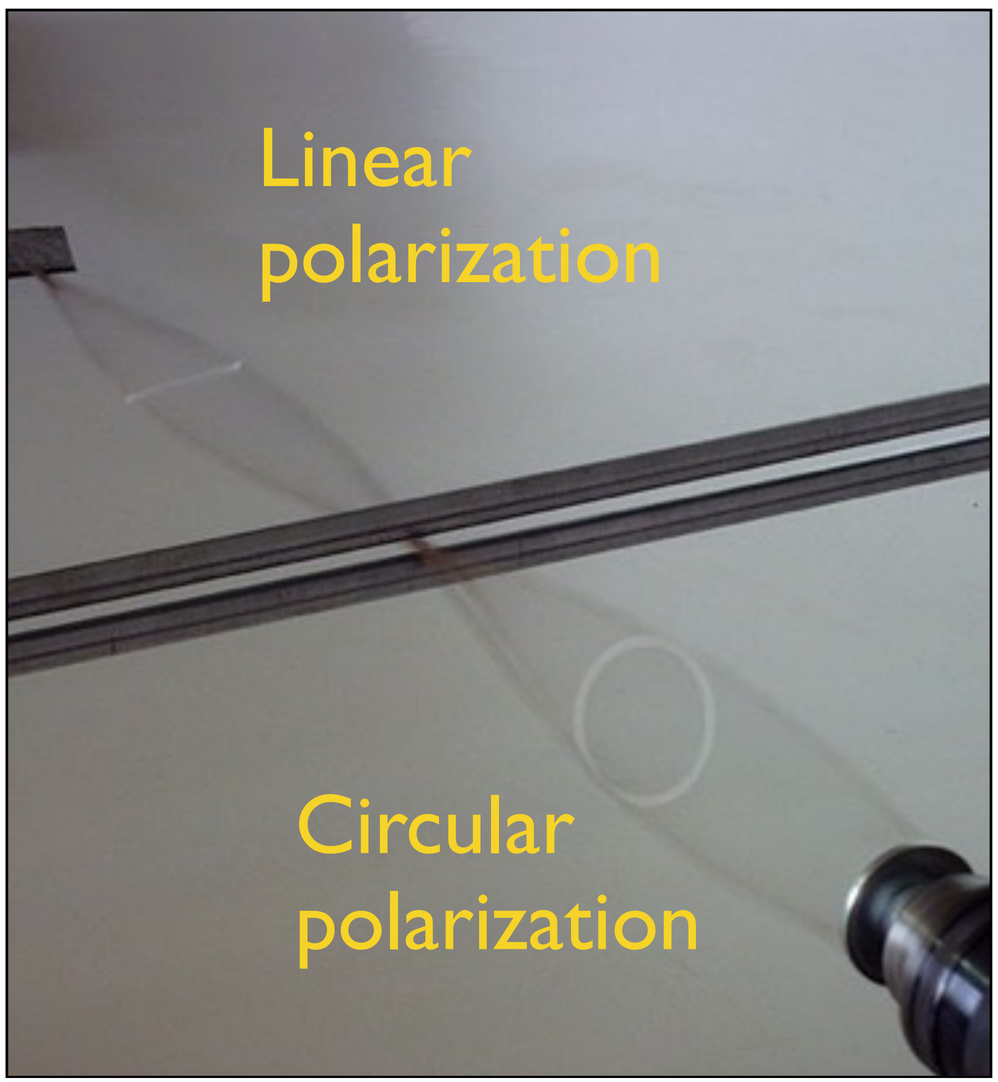
- Wavelength: λ, length to repeat peak-peak (trough-trough)
- Period: τ, time to repeat one cycle of the wave (seconds)
- Phase: position within the wave cycle (a.k.a. *phase shift* or *offset*)
- Frequency: $f = 1/\tau$, units of 1/sec (Hertz). Also: $\omega = 2\pi f = 2\pi/\tau$
- Wavenumber: $k = 2\pi/\lambda$, in units of 1/meter ("spatial frequency")
- Velocity: $v = \lambda f$, in units of length/time
- Amplitude: A. Energy: $E \sim (\text{Amplitude})^2$

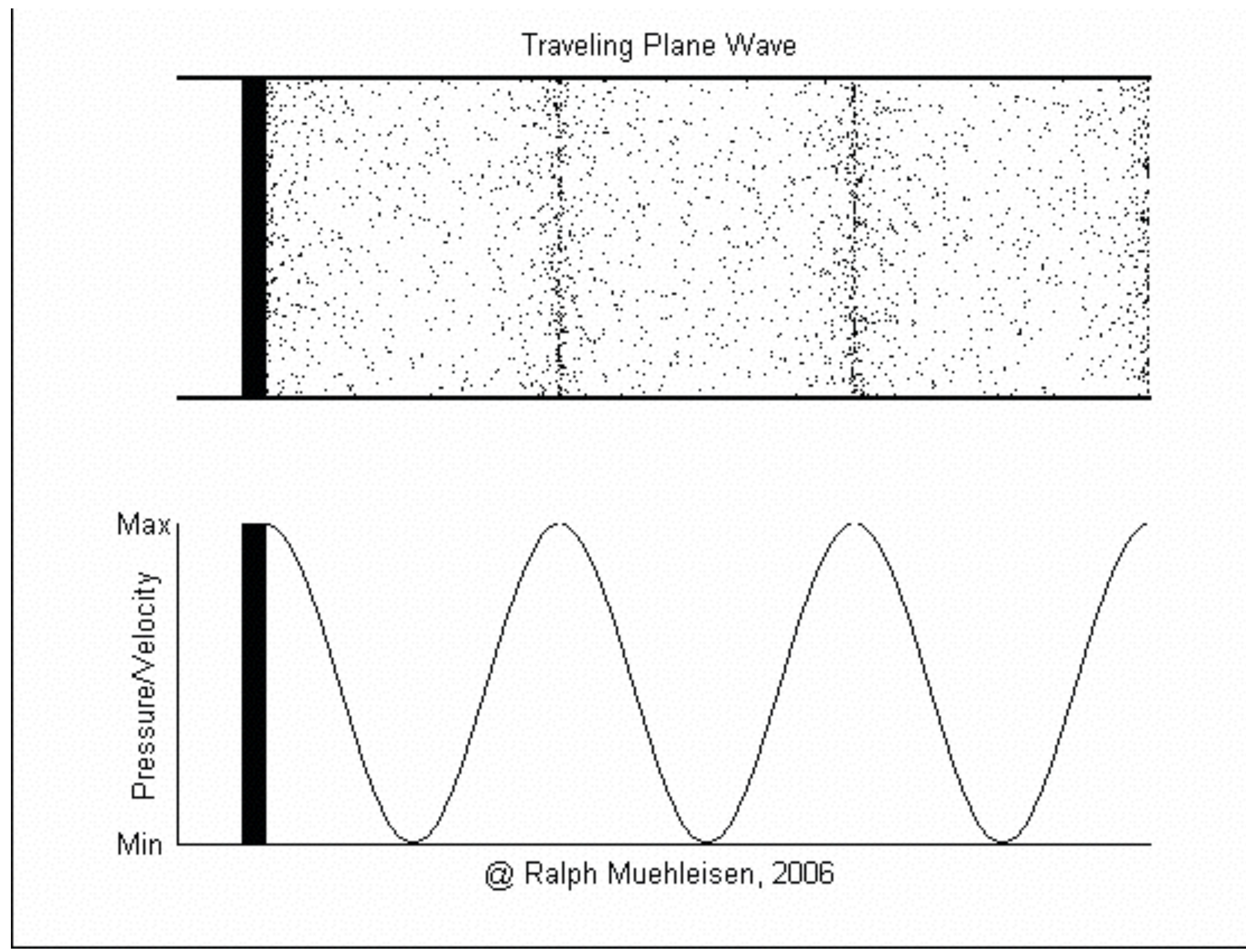
Behavior of Waves

- Behavior typical of waves:
  - Reflection: a wave strikes a surface and bounces off
  - Refraction: when a wave changes direction after passing between two media of different densities
  - Diffraction: the bending and spreading of waves around an obstacle, often creating an *interference* pattern
  - Polarization: the orientation of the oscillation of transverse waves
- Polarization is not important in acoustics. Why is that?

Transverse & Longitudinal Waves

- Sound waves are longitudinal pressure waves; oscillation occurs *along* the direction of propagation





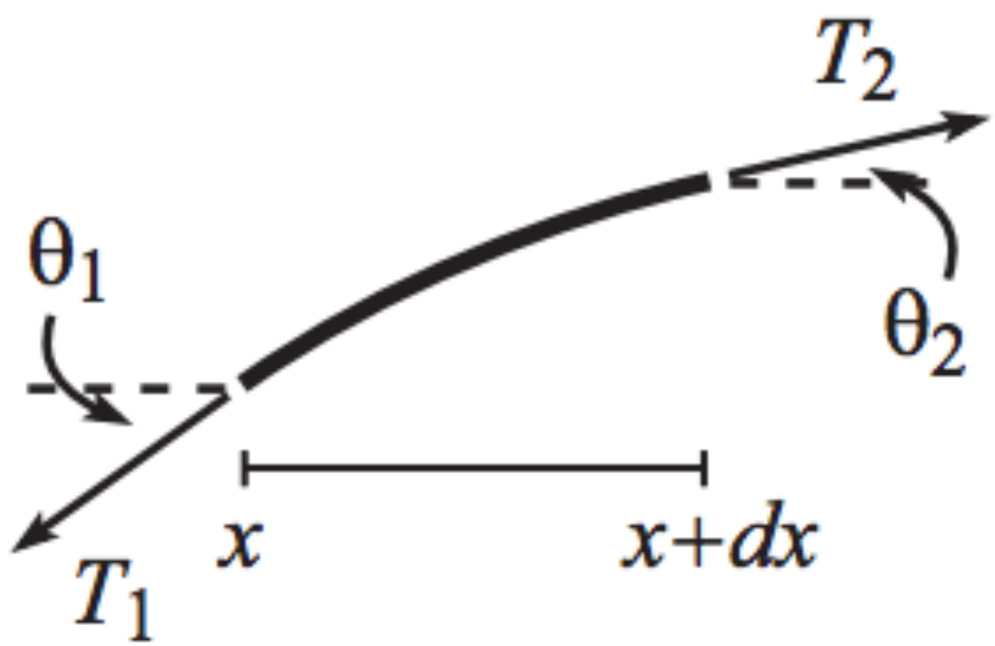
Waves on a String

- Suppose we have a rope of length *L*, and *L* is so long that, for now, we don't worry about the ends flopping around
- We shake and vibrate the rope, sending pulses traveling down its length

- What are the properties of the wave on this rope? It's speed, its wavelength, etc.?

Waves on a String

- Imagine a little piece of the string with length *dx*. It's under tension, i.e., it feels pulling forces T_1 and T_2 at each end that try to move the piece up or down



$$\sum F_y = ma_y = T_{2y} - T_{1y}$$
$$= T_2 \sin\theta_2 - T_1 \sin\theta_1$$

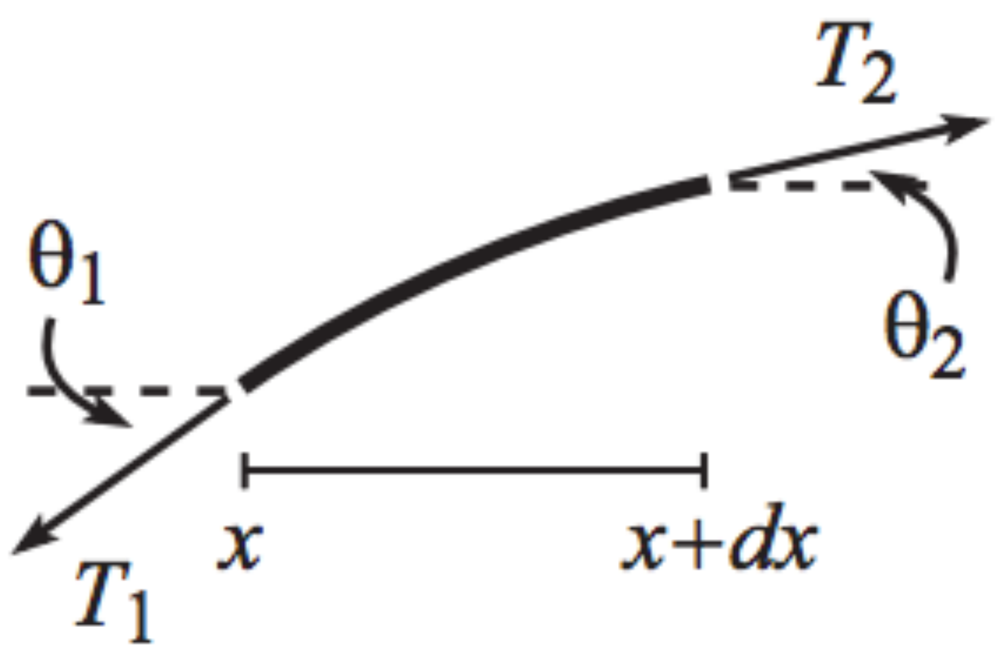
← *Newton's 2nd Law*: force on piece of rope with mass *m*

Assumptions Made

- Angle $\theta_1 \sim \theta_2$, which means the tension T on each side of the piece is approximately the same
- The mass of the piece m is really small, so the effect of gravity ($F = mg$) is negligible compared to T
- Also note:
  - The total mass of the rope is M and its length is L
  - The *mass density* of the rope is $\rho=M/L$, in units of mass per unit length (e.g., g/cm)
  - So the mass of the piece is $m = \rho\ dx$

Waves on a String

- We also need to sum forces in the *x* direction:



$$\sum F_x = ma_x = T_{2x} - T_{1x}$$
$$= T_2 \cos\theta_1 - T_1 \cos\theta_2$$
$$\approx T - T$$
$$= 0$$

Forces along *x* direction sum to zero; the piece of rope doesn't move side-to-side

The Wave Equation

- With a few more substitutions (see overflow slides) Newton's second law reduces to the expression

$$\frac{d^2y}{dt^2} = \frac{T}{\rho}\cdot\frac{d^2y}{dx^2} = v^2\frac{d^2y}{dt^2}, \qquad \text{where } \; v = \sqrt{\frac{T}{\rho}}$$

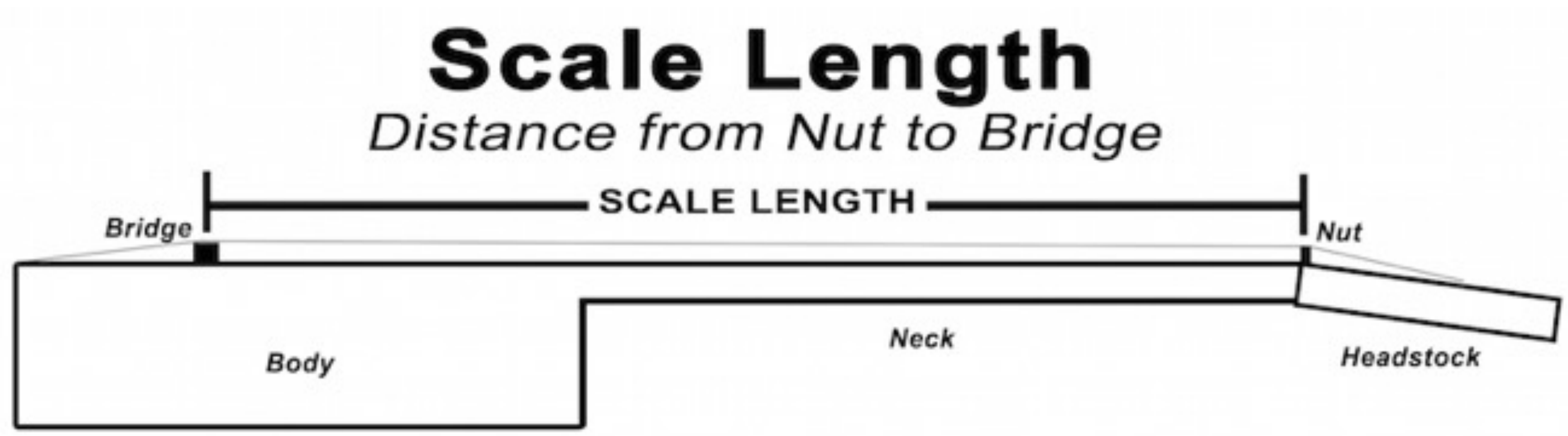
- This is the wave equation that describes the motion of the piece of rope vs. time *t* and position *x*. It has two solutions:

$$\begin{aligned} y(x,t) &= A\sin(kx \pm \omega t) \\ &= A\sin\frac{2\pi}{\lambda}(x \pm vt), \qquad \text{where } v = \lambda f = \sqrt{\frac{T}{\rho}} \end{aligned}$$

- Traveling waves, depend on physical properties of the rope

A Vibrating String

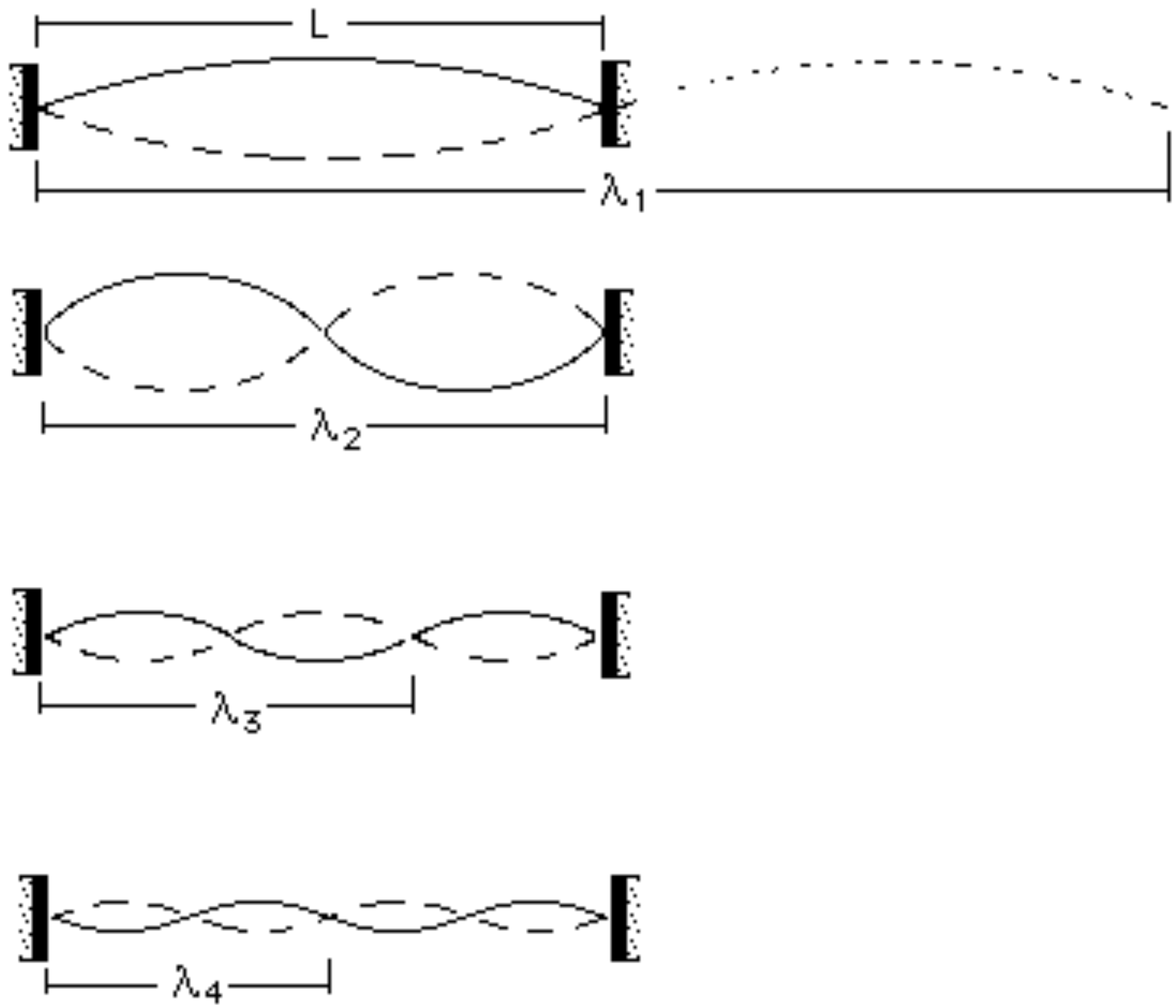
- In a musical instrument with a vibrating string, the endpoints are fixed so that they don't vibrate
- Example: a guitar string is fixed at the nut and bridge and will not vibrate at those points



- What does the wave on the string look like in this case?

The Plucked String

- If the string is fixed at both ends, it's going to look something like this when you pluck it:



$L = \lambda_1/2$

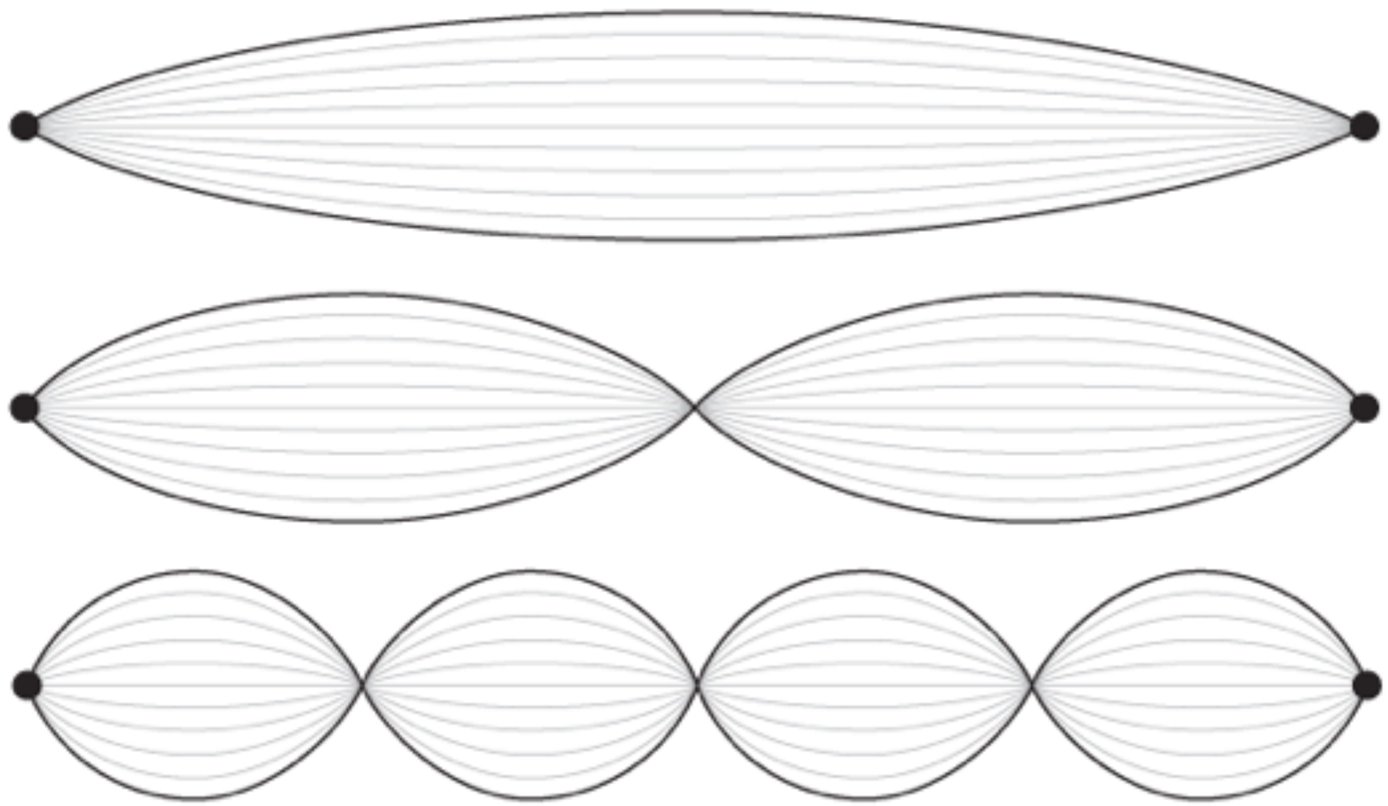
$L = \lambda_2$

$L = 3\lambda_3/2$

$L = 2\lambda_4$

Standing Waves

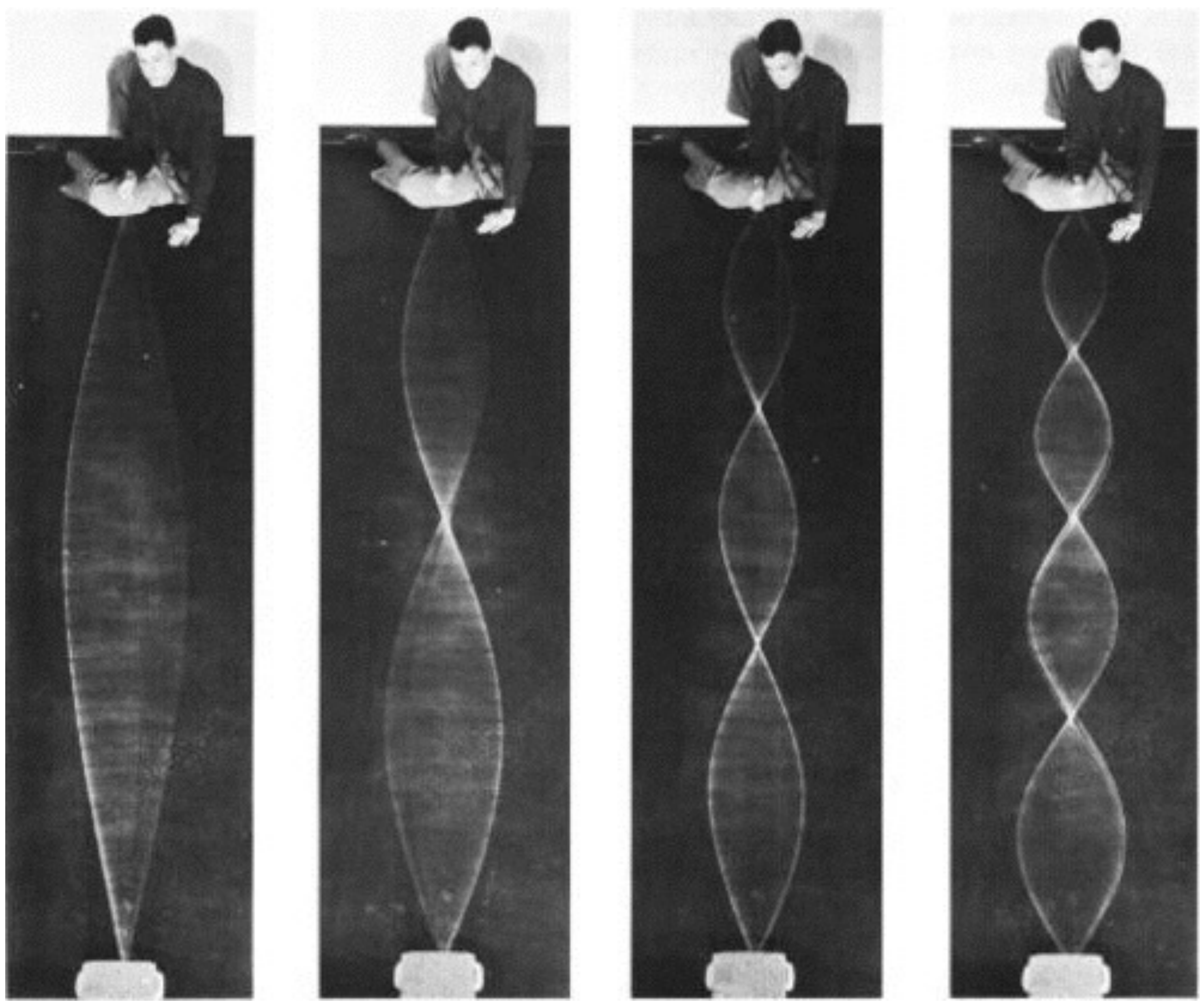
- These patterns are called standing waves



- You can construct a standing wave from a superimposed combination of traveling waves moving in both directions
- So our earlier conclusions ($v = \lambda f = \sqrt{T/\rho}$) are still valid and can be used to describe the fixed string!

Producing Standing Waves

- We can create large standing waves in a string by driving it with an oscillating motor



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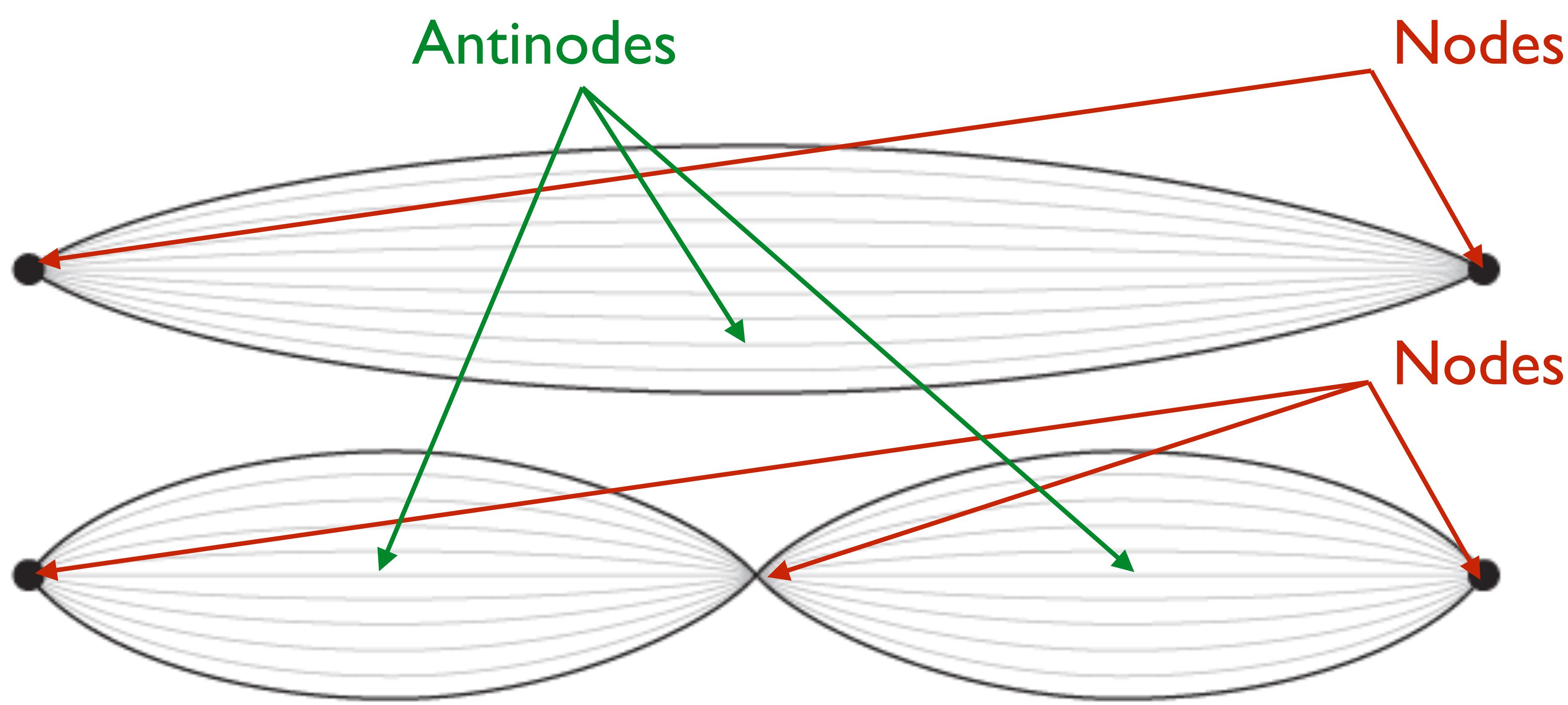
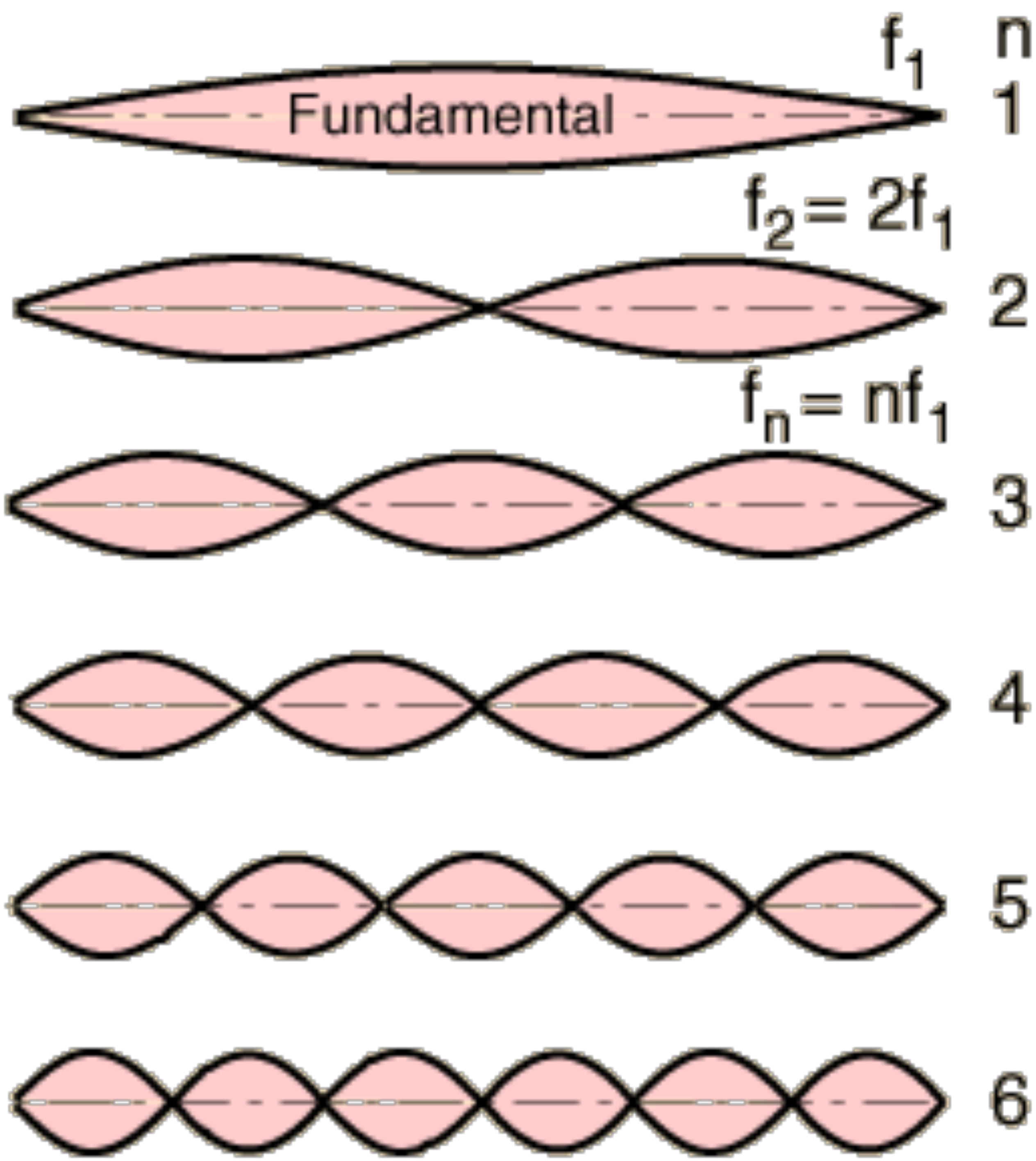
Terminology

- Nodes: points where the string is fixed (or held) and cannot vibrate
- Antinodes: points of strongest vibration/oscillation along the length of the string



Harmonics/Overtones



$L = \lambda/2$ $\quad$ $f_1 = v / \lambda = v/2L$

$L = \lambda$ $\quad$ $f_2 = v/L = 2f_1$

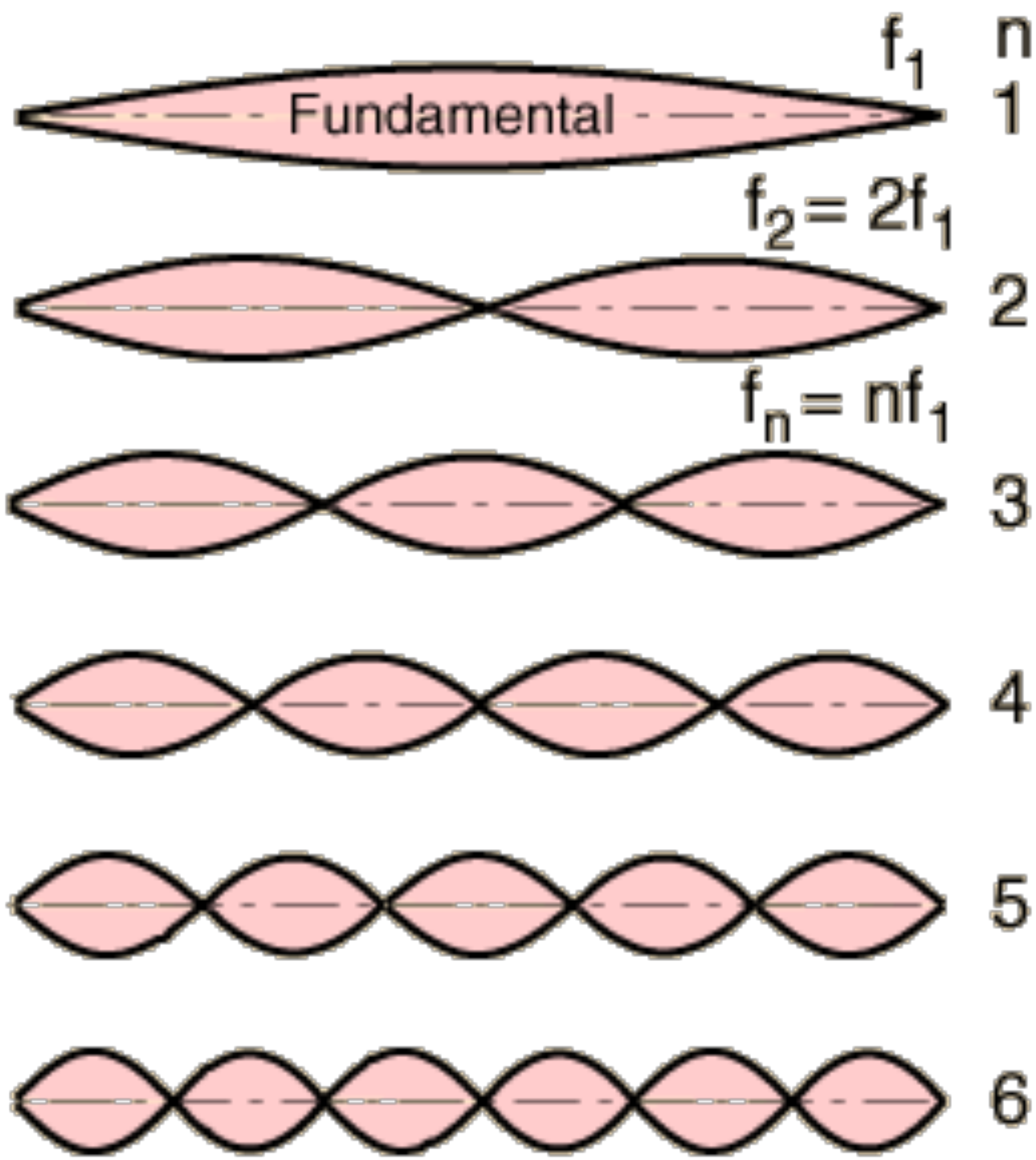
$L = 3\lambda/2$ $\quad$ $f_3 = 3v/2L = 3f_1$

$L = 2\lambda$ $\quad$ $f_4 = 2v/L = 4f_1$

$L = 5\lambda/2$ $\quad$ $f_5 = 5v/2L = 5f_1$

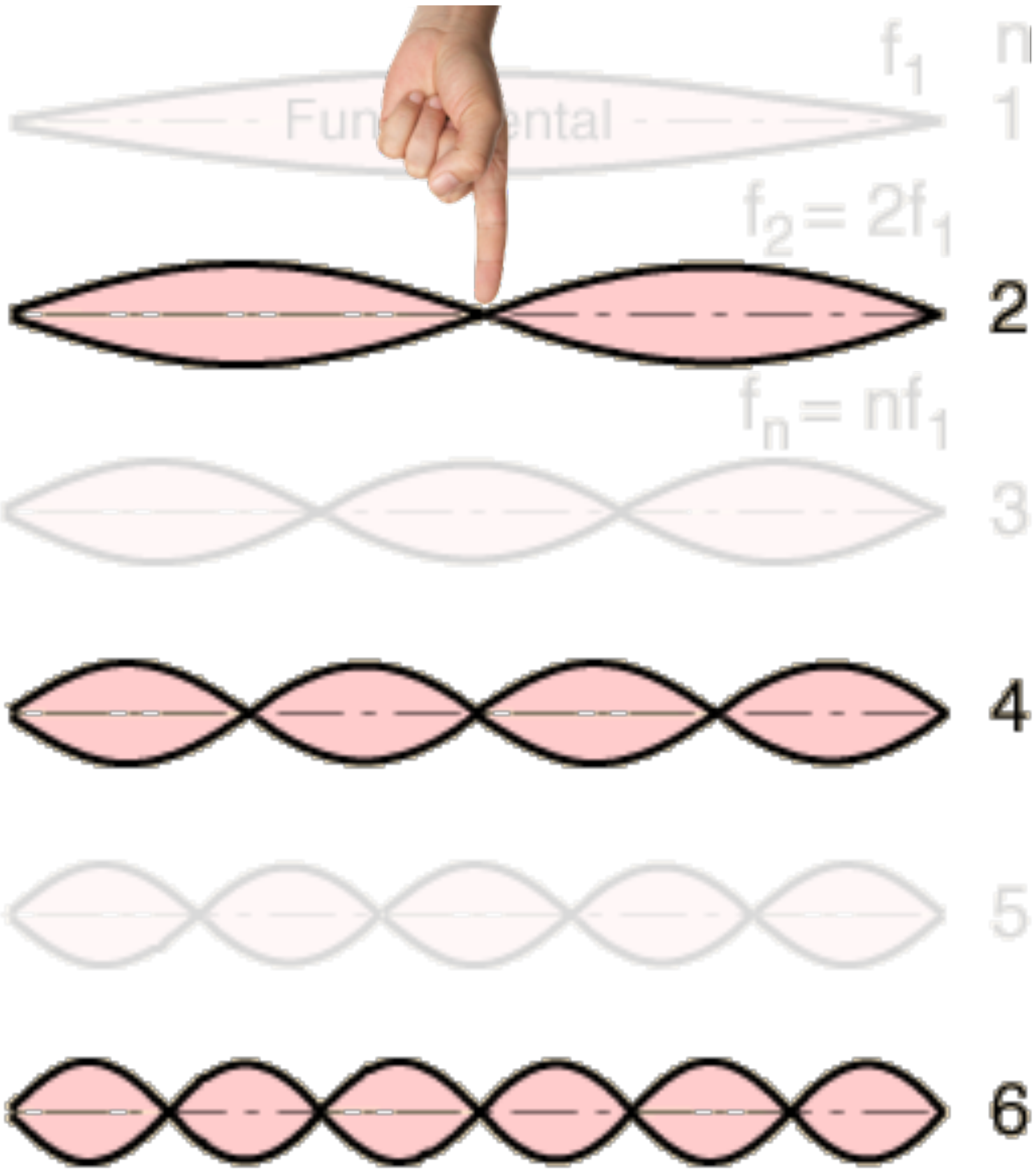
$L = 3\lambda$ $\quad$ $f_6 = 3v/L = 6f_1$

Harmonics/Overtones



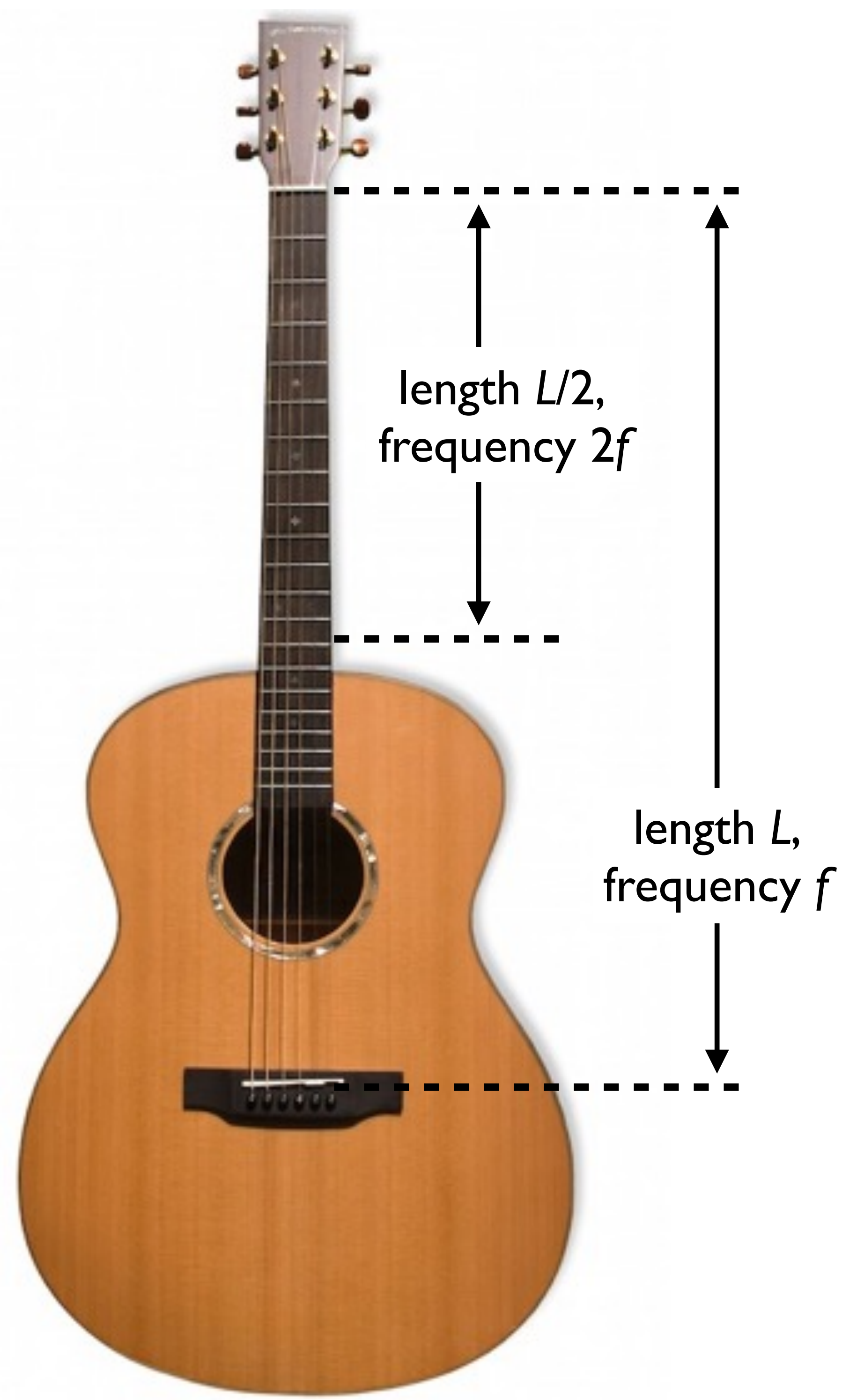
- An open string will vibrate in its fundamental mode *and* overtones at the same time
- True not just for strings, but all vibrating objects
- We will demonstrate the presence of overtones by making a spectrogram of a plucked string

Harmonics/Overtones



- If a string is touched at its midpoint, it can only vibrate at frequencies with a node at the midpoint
- The odd-integer harmonics (including the fundamental frequency) are suppressed
- Question: what will the note sound like?

Notes and String Length



- Mathematical relationship between string length and pitch
- When you halve the string, the pitch goes up by one octave
- Cutting the string in half means the frequency goes up by 2
- One octave = doubling of the frequency of the note
- Let's try it out with a couple of monochords…

Simple Harp

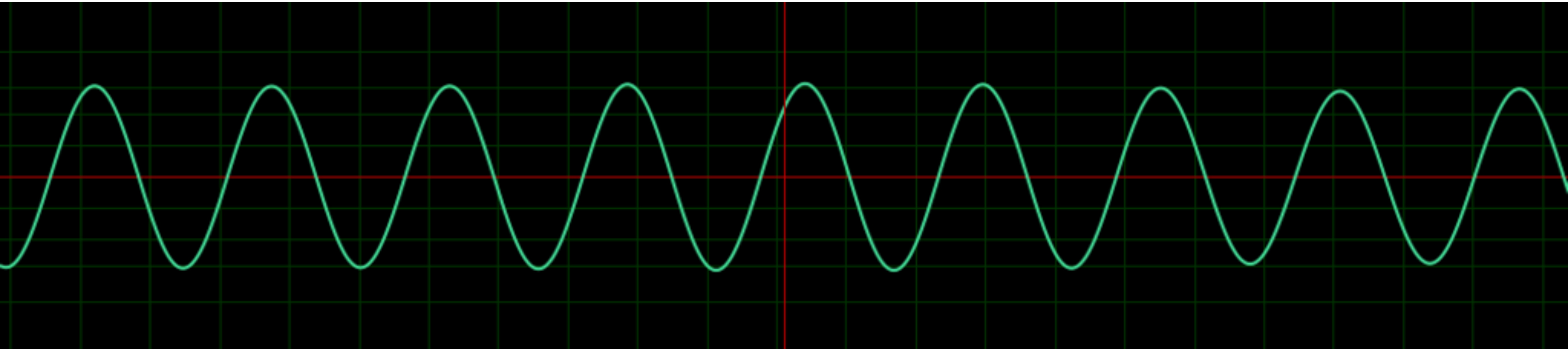
- Music Maker "lap harp" for teaching music to children
- Very simple layout with 9 identical strings
- Question: does the string length drop by half as we go up in octaves? Let's measure it…
- Remember: $f_1 = v/\lambda_1 = \sqrt{(T/\rho)}/2L$
- String tension (and density) matter as well as length!

Piano Strings

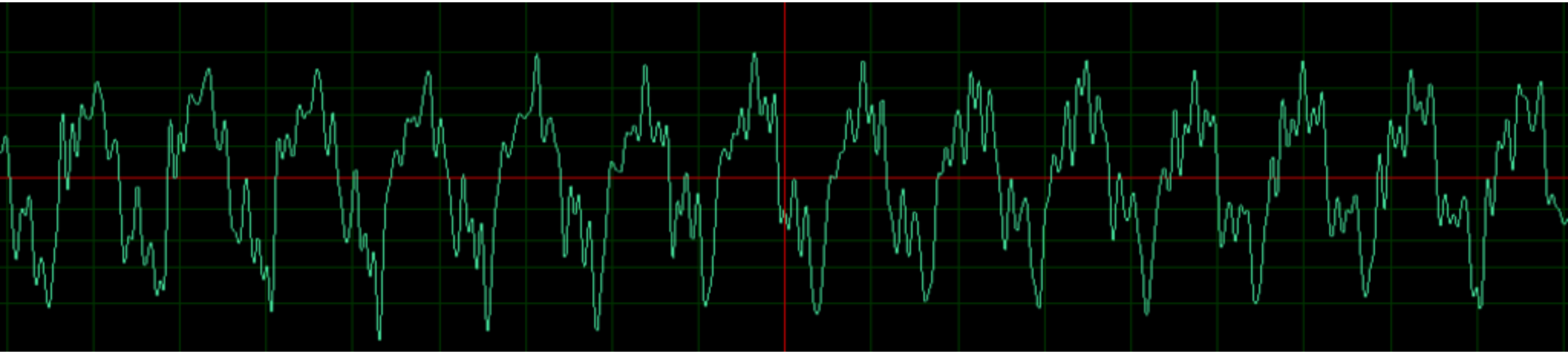
- Instrument makers take advantage of the dependence of f on T and ρ as well as L
- About 20T of tension (all strings combined) in a grand piano
- Note: the bass strings are much thicker and denser than the treble strings
- Otherwise, the frame would need to be 100s of feet long

Playing the Harp

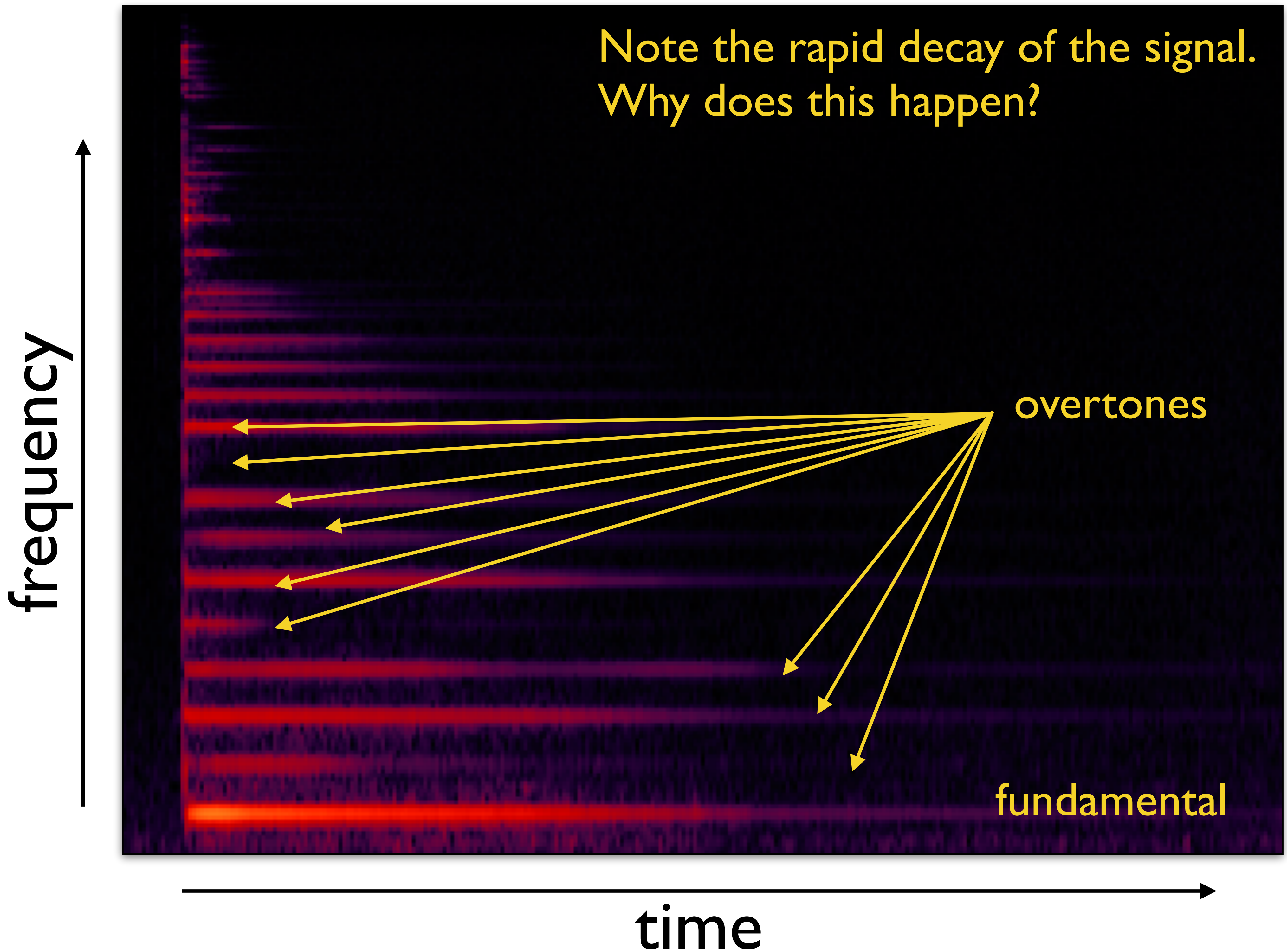
- If we pluck **G4**, what do you expect to observe?



- In fact, this is the true waveform:

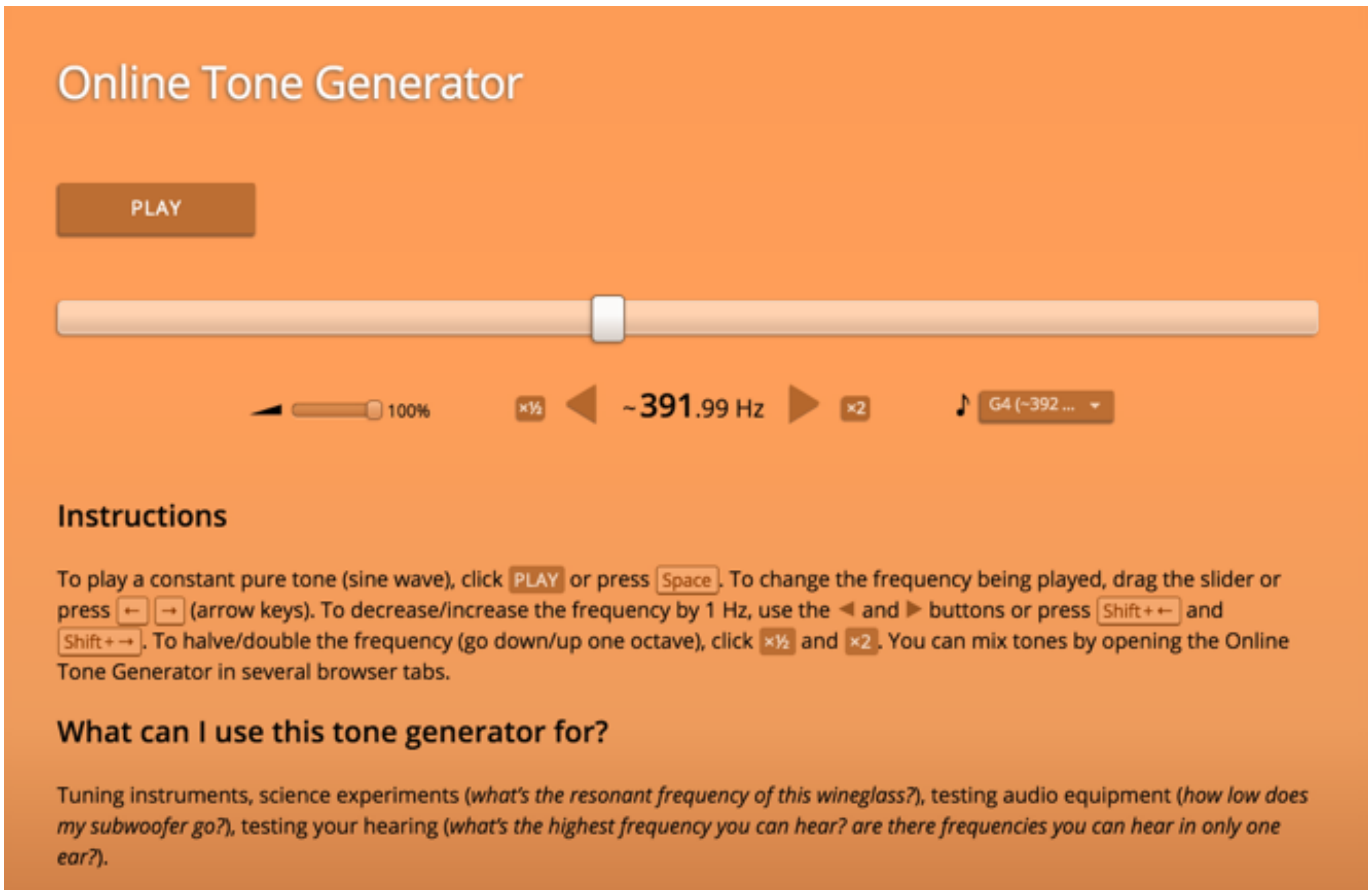


Spectrogram of the Harp



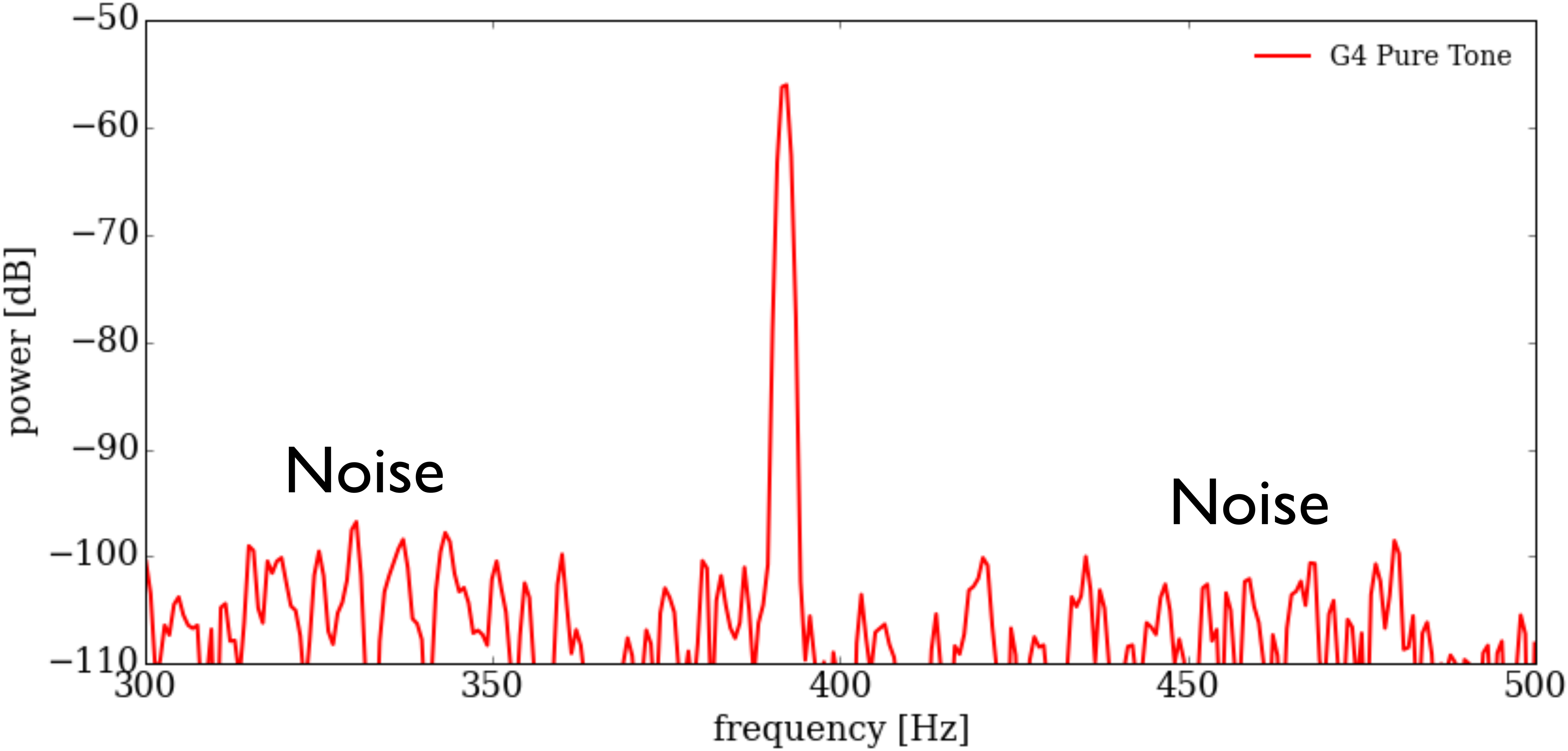
Making Pure Tones

- If you don't have an open speaker and function generator, you can go here:
  - http://plasticity.szynalski.com/tone-generator.htm



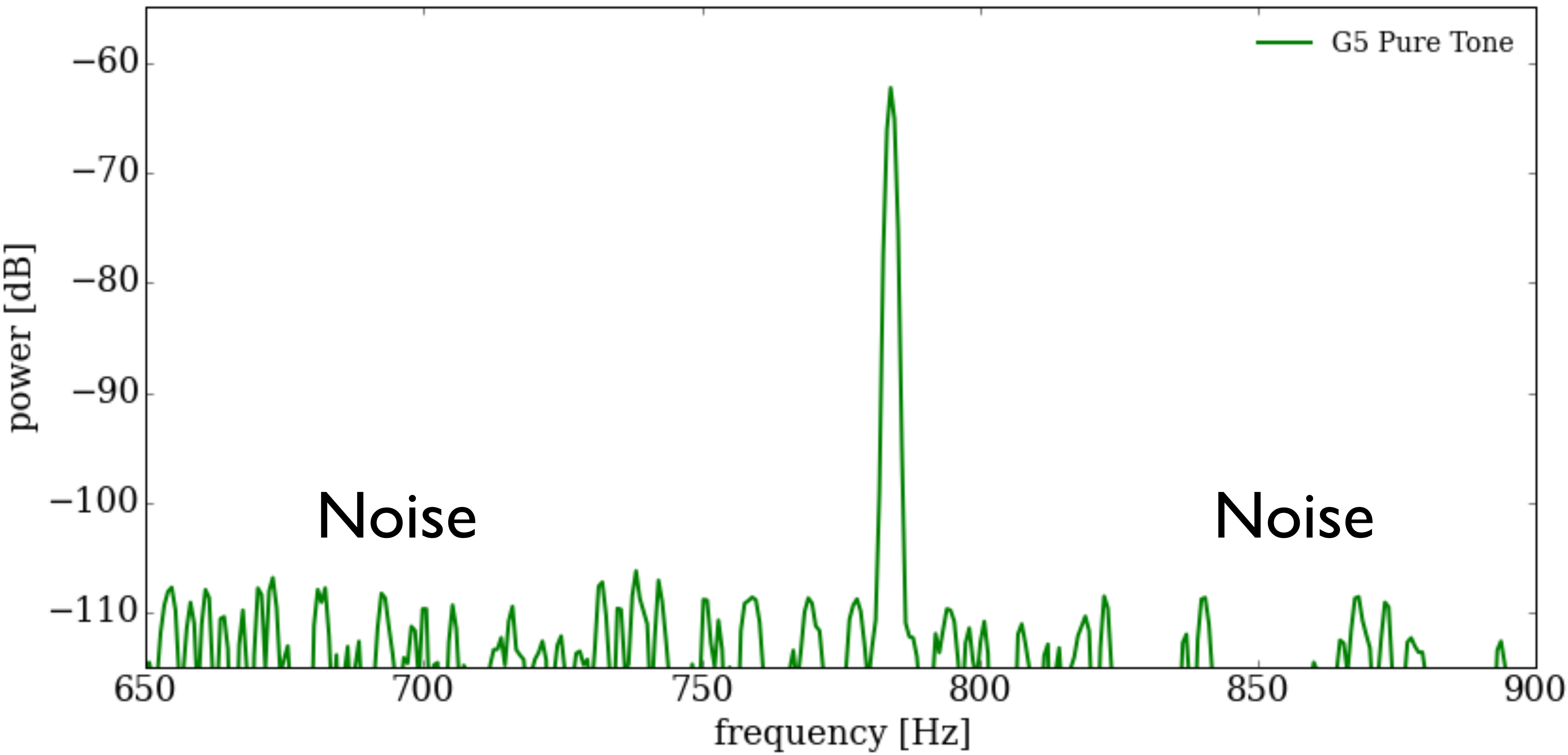
Spectrum of a Pure Tone

- Pure sine wave looks like a spike at one frequency



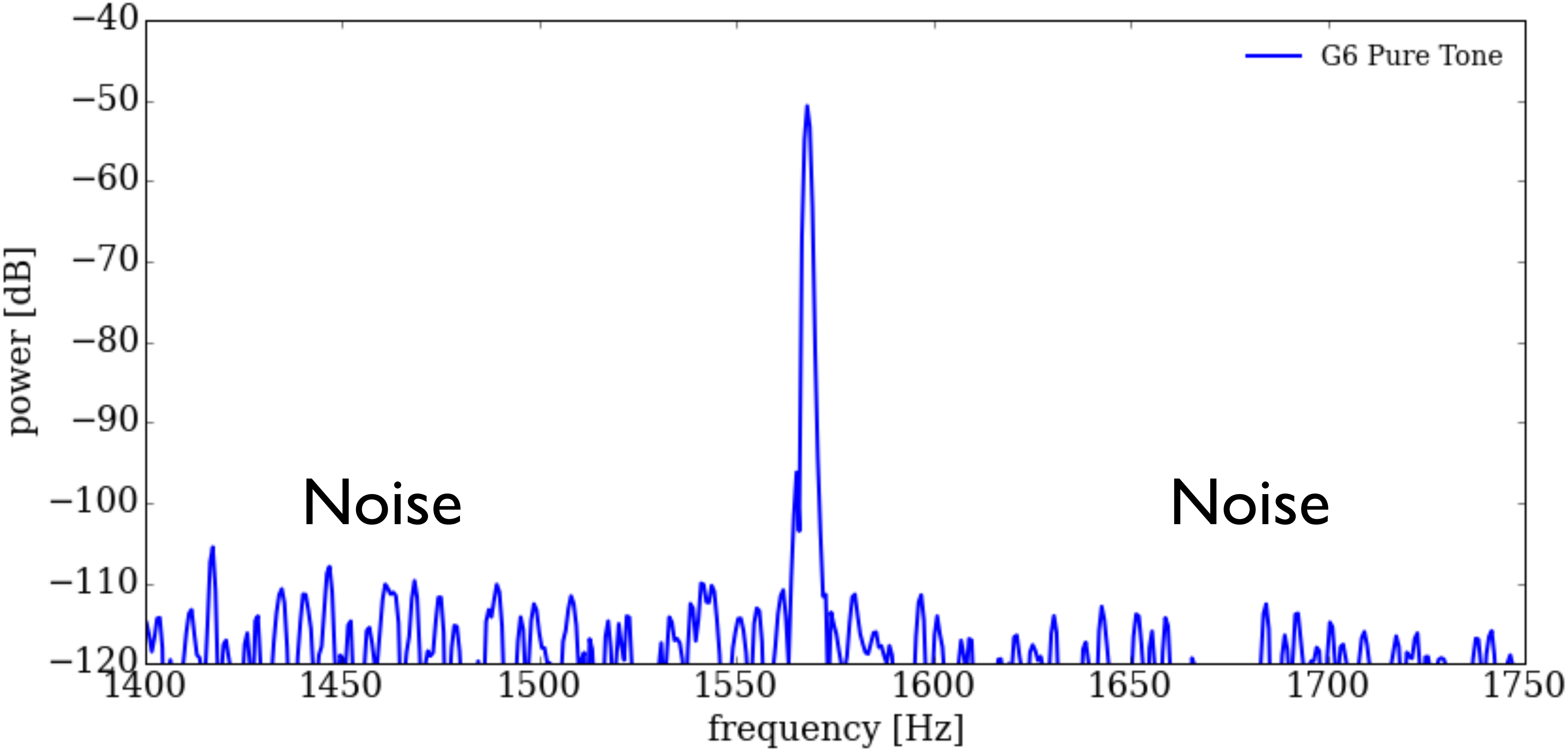
Spectrum of Pure G5

- Pure sine wave looks like a spike at one frequency



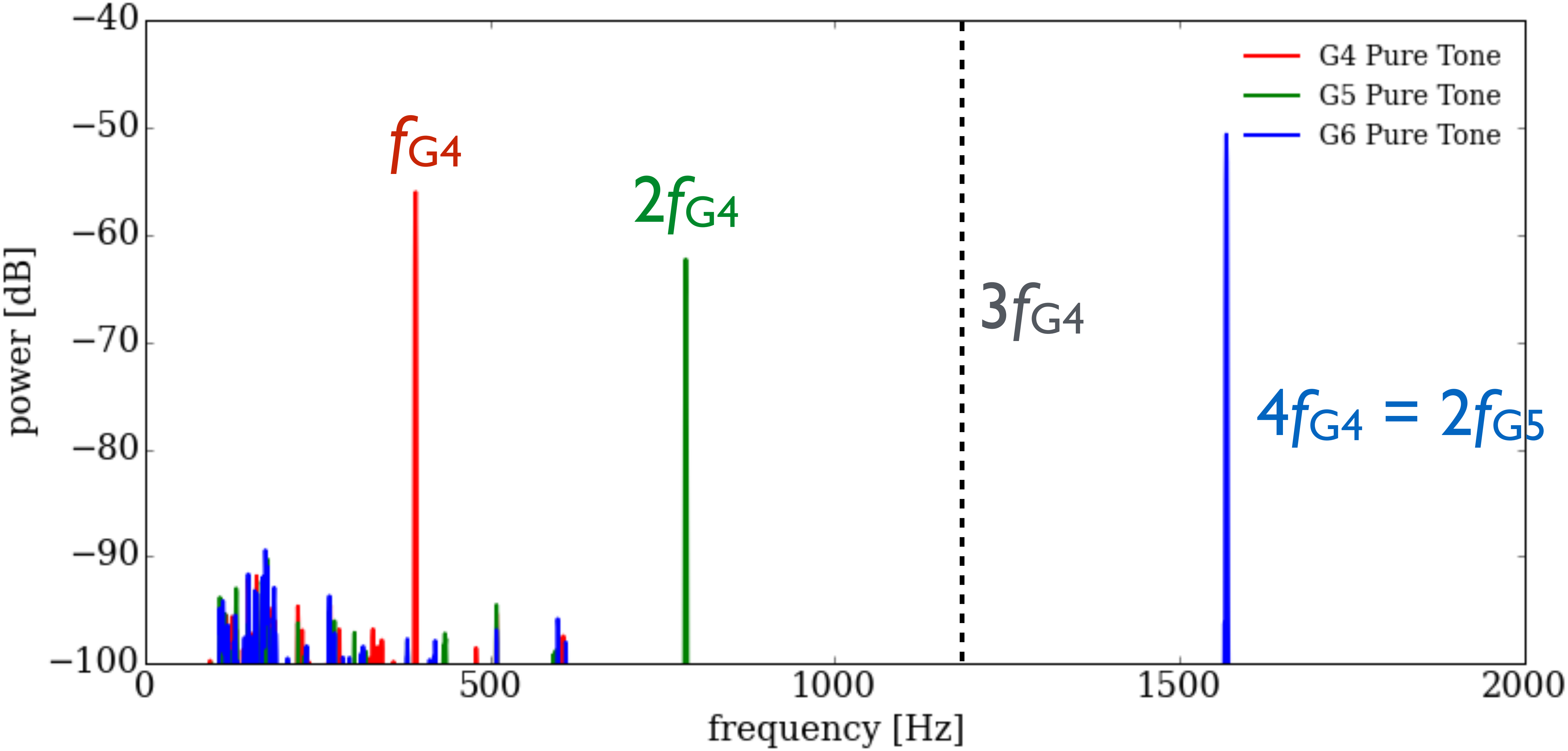
Spectrum of Pure G6

- Pure sine wave looks like a spike at one frequency

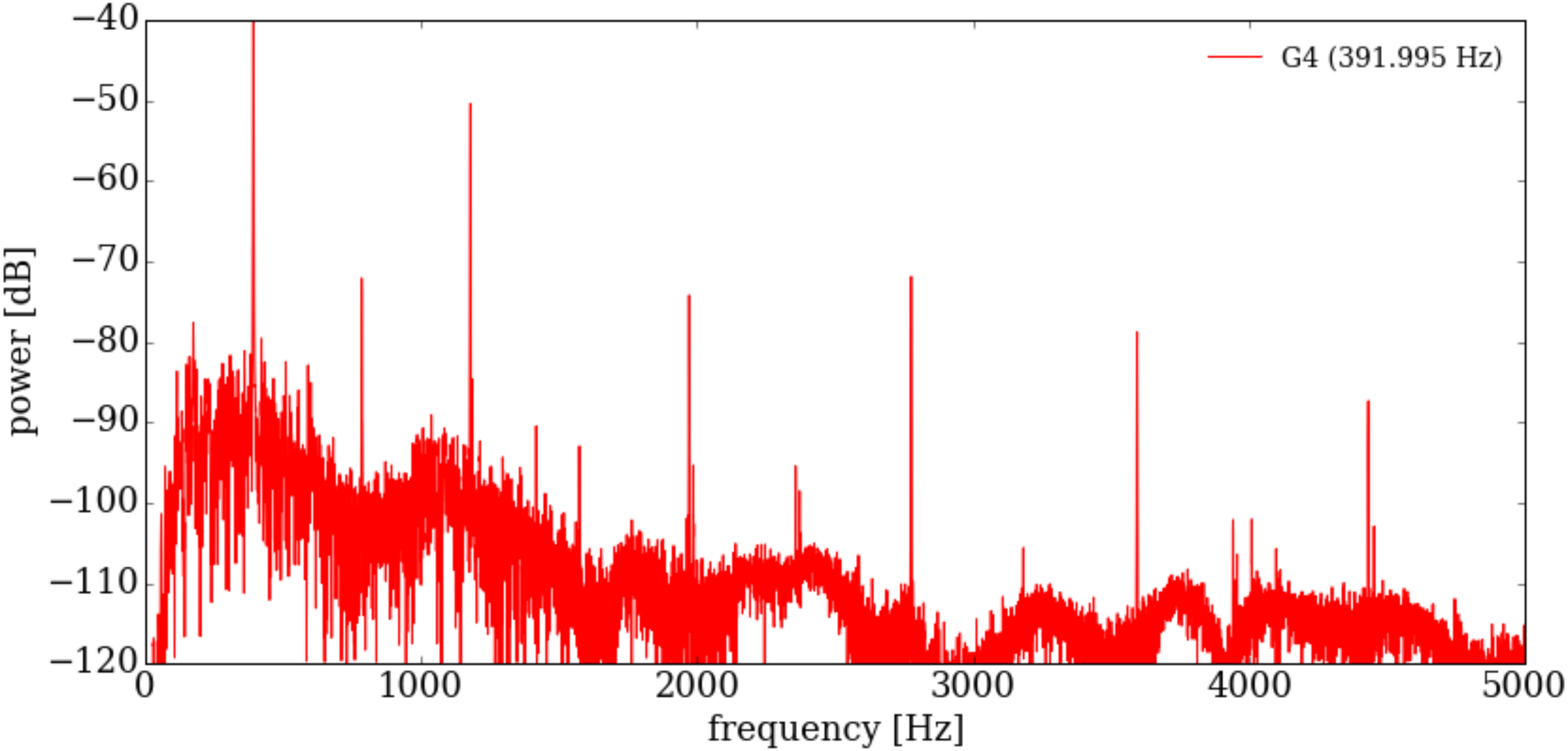


Pure G4, G5, G6

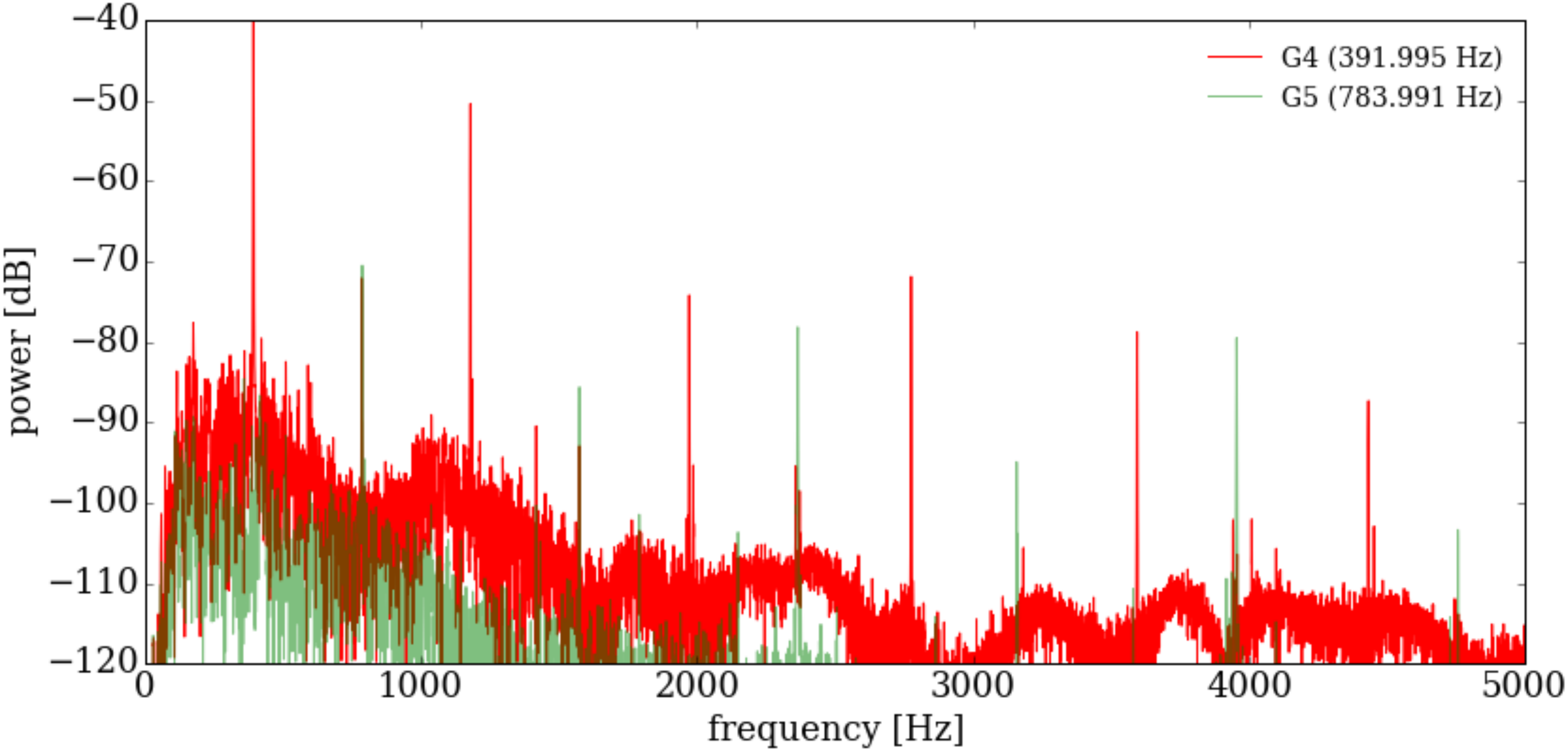
- Note the integer relationship between the pure tones



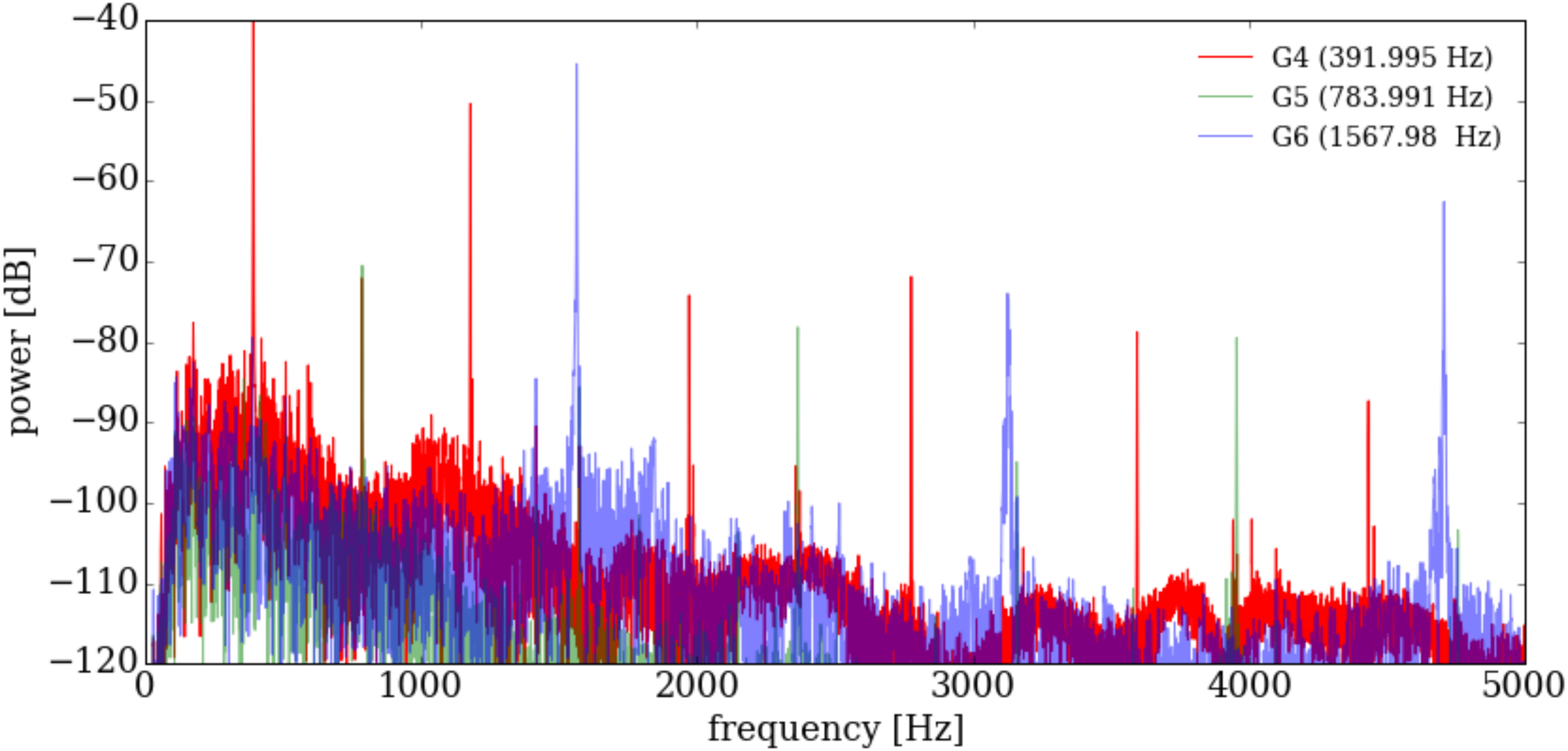
Power Spectrum of G4



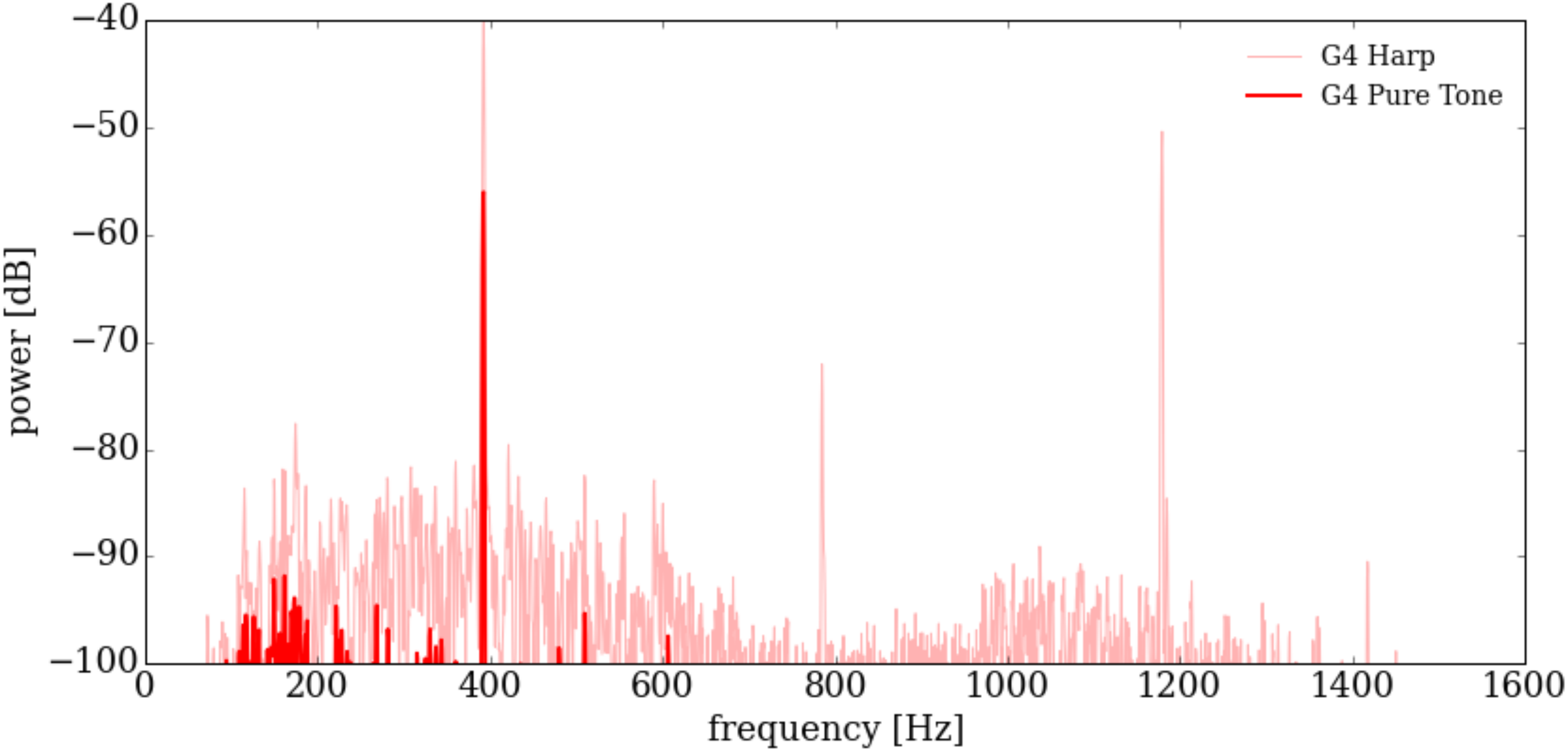
Spectrum of G4 and G5



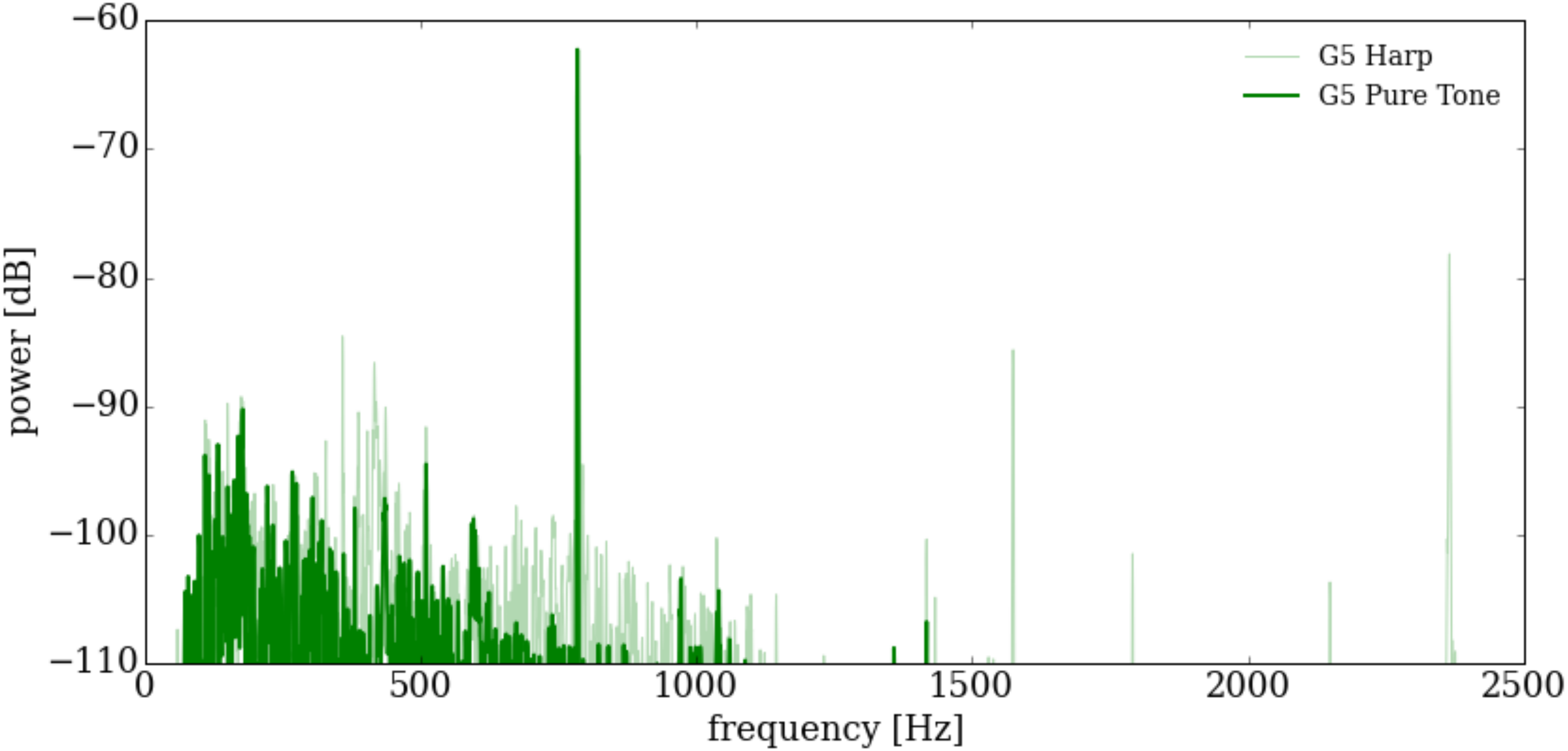
Spectrum of G4, G5, & G6



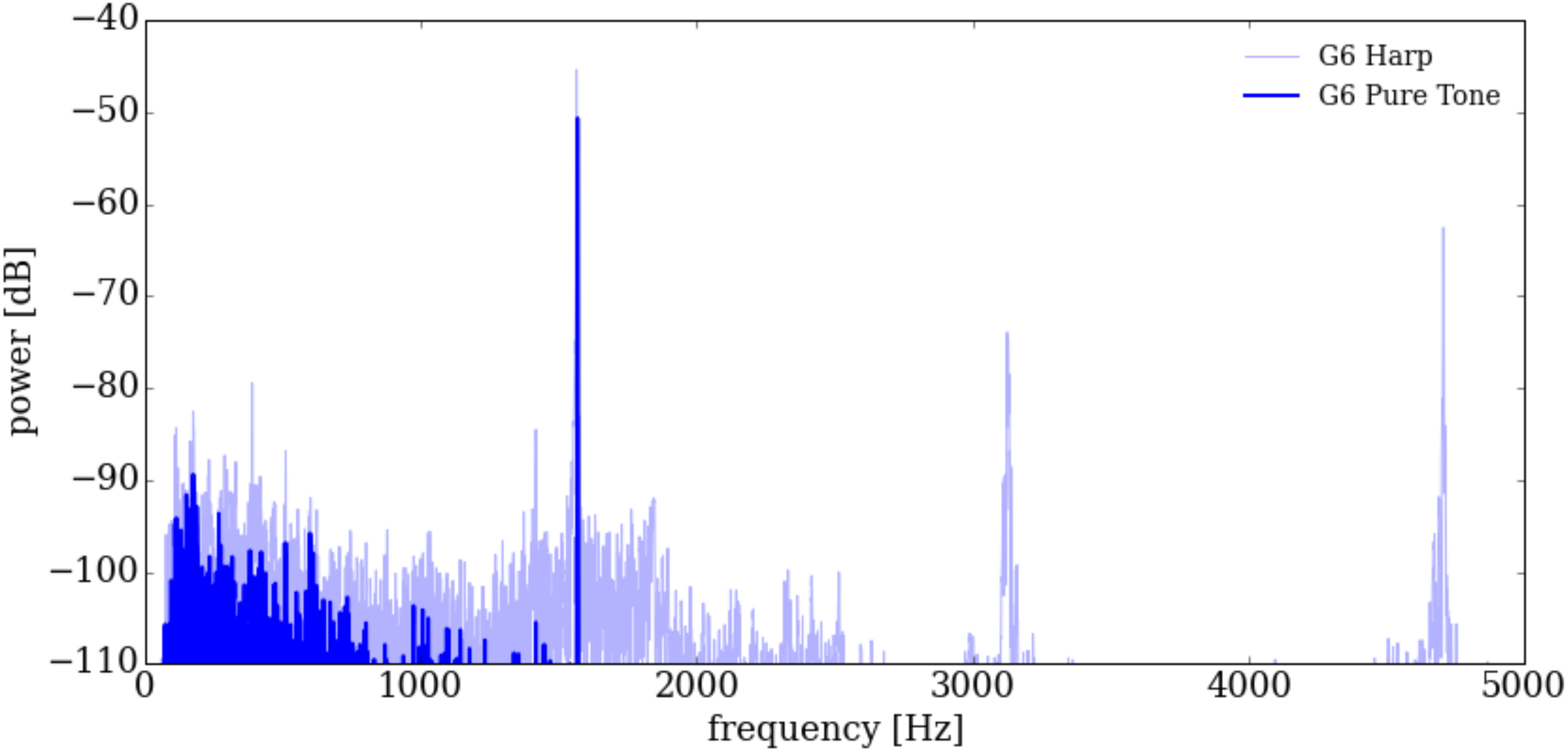
Harp and Pure Tone: G4



Harp and Pure Tone: G5

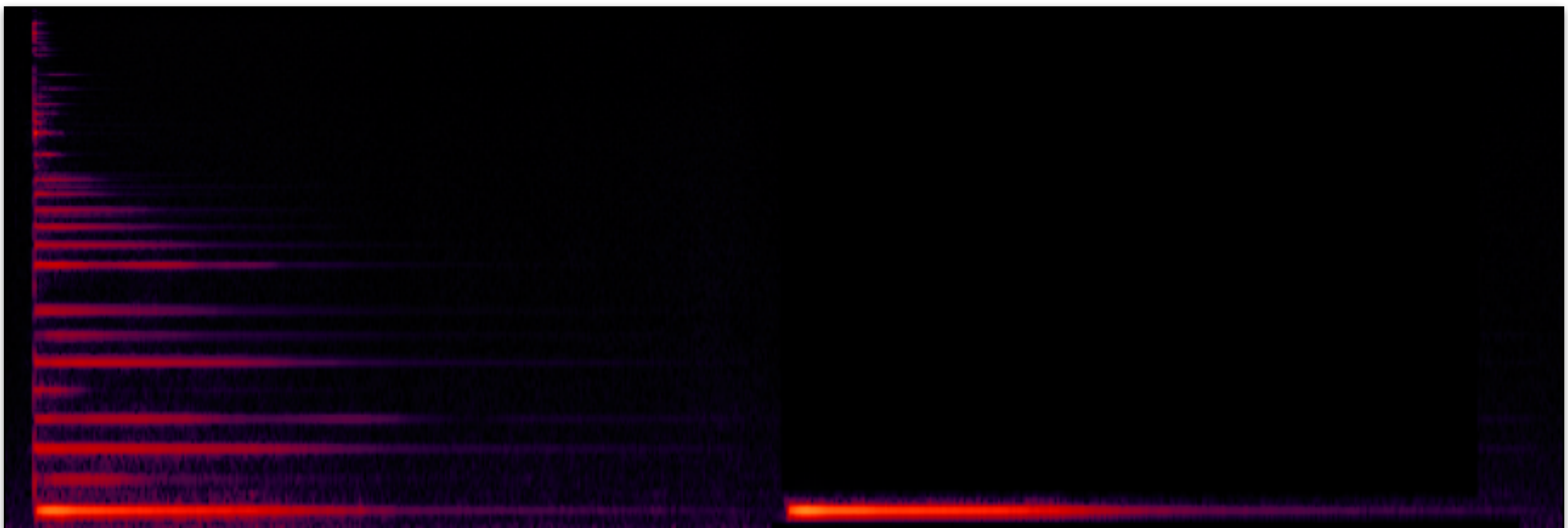


Harp and Pure Tone: G5



"Cleaning" the Spectrogram

- We can use Audition to remove the overtones from the second "pluck" in the spectrogram

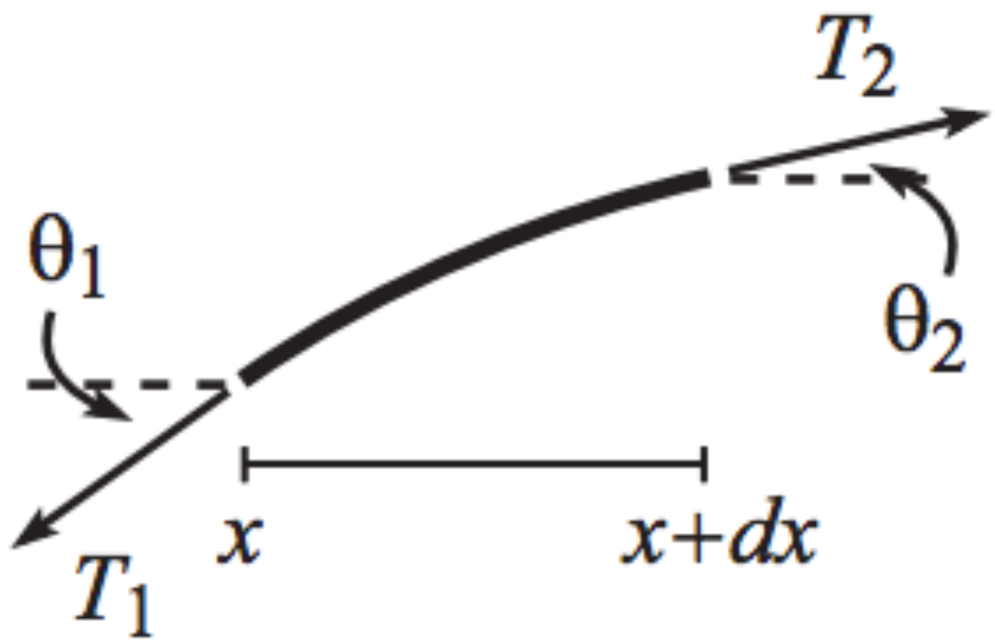


- What do you think the second pluck will sound like after cleaning?

Summary

- Waves on a string move with velocity $v = \sqrt{T/\rho}$
  - T is the string tension and ρ is the density
- Open strings fixed at both ends will exhibit standing waves
  - Increasing number of higher harmonics or overtones
  - Integer multiples of fundamental tone with $f_1=\sqrt{(T/\rho)}/2L$
  - Nodes: positions where the string doesn't oscillate
  - Antinodes: positions of maximum oscillation
- When a string is plucked or driven, all of the overtones can be excited simultaneously. But only some are dominant and determine the timbre

Wave on a Rope: Geometry



$$ma_y = T_{2,y} - T_{1,y}$$
forces on rope segment

$$m\frac{d^2y}{dt^2} = T_2 \sin\theta_2 - T_1 \sin\theta_1$$
rewrite in terms of angles

$$\rho \cdot dx \frac{d^2y}{dt^2} \approx T\sin\theta_2 - T\sin\theta_1$$
rewrite *m* = ρ *dx*, note that *T* is the same on both ends

$$\approx T(\tan\theta_2 - \tan\theta_1)$$
if *θ* small, sine ~ tangent

The Wave Equation

$$\rho \cdot dx \frac{d^2 y}{dt^2} \approx T(\tan\theta_2 - \tan\theta_1)$$

$$= T\left(\left.\frac{dy}{dx}\right|_2 - \left.\frac{dy}{dx}\right|_1 \right)$$

definition of tangent

$$\frac{d^2 y}{dt^2} = \frac{T}{\rho}\left(\left.\frac{dy}{dx}\right|_2 - \left.\frac{dy}{dx}\right|_1 \right) \Big/ dx$$

group terms on right side

$$\frac{d^2 y}{dt^2} = \frac{T}{\rho} \frac{d^2 y}{dx^2}$$

definition of d^2y/dx^2

$$\boxed{\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2}}$$

define $v^2 = T / \rho$