

PHY 103: Scales and Musical Temperament

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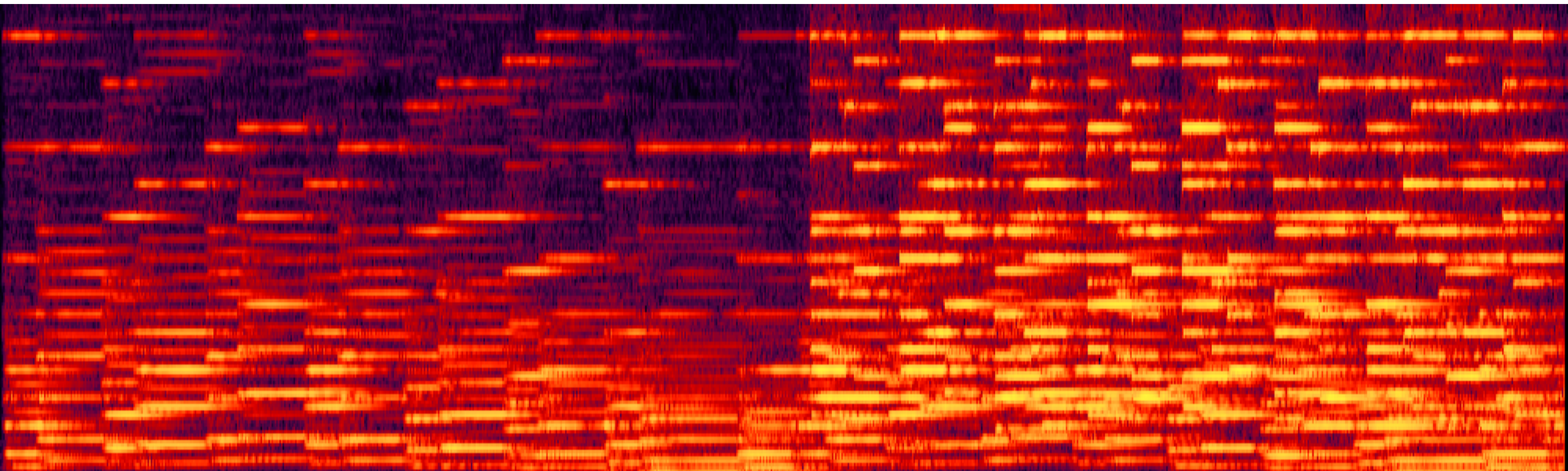
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Reading

- ▶ Hopkin Chapter 3
- ▶ Berg and Stork Chapter 9

Musical Structure

- ▶ We've talked a lot about the physics of producing sounds in instruments
- ▶ We can build instruments to play **any** fundamental tone and overtone series



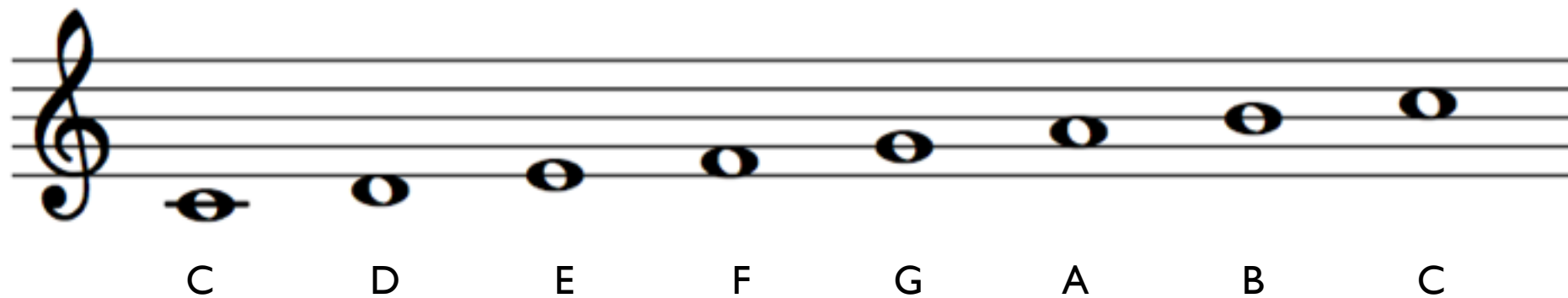
Scales

- ▶ In practice, we don't do that. There are agreed-upon conventions for how notes are supposed to sound

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- ▶ Why is that? How did those conventions come about? Is there a reason for it?
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What's a Scale?

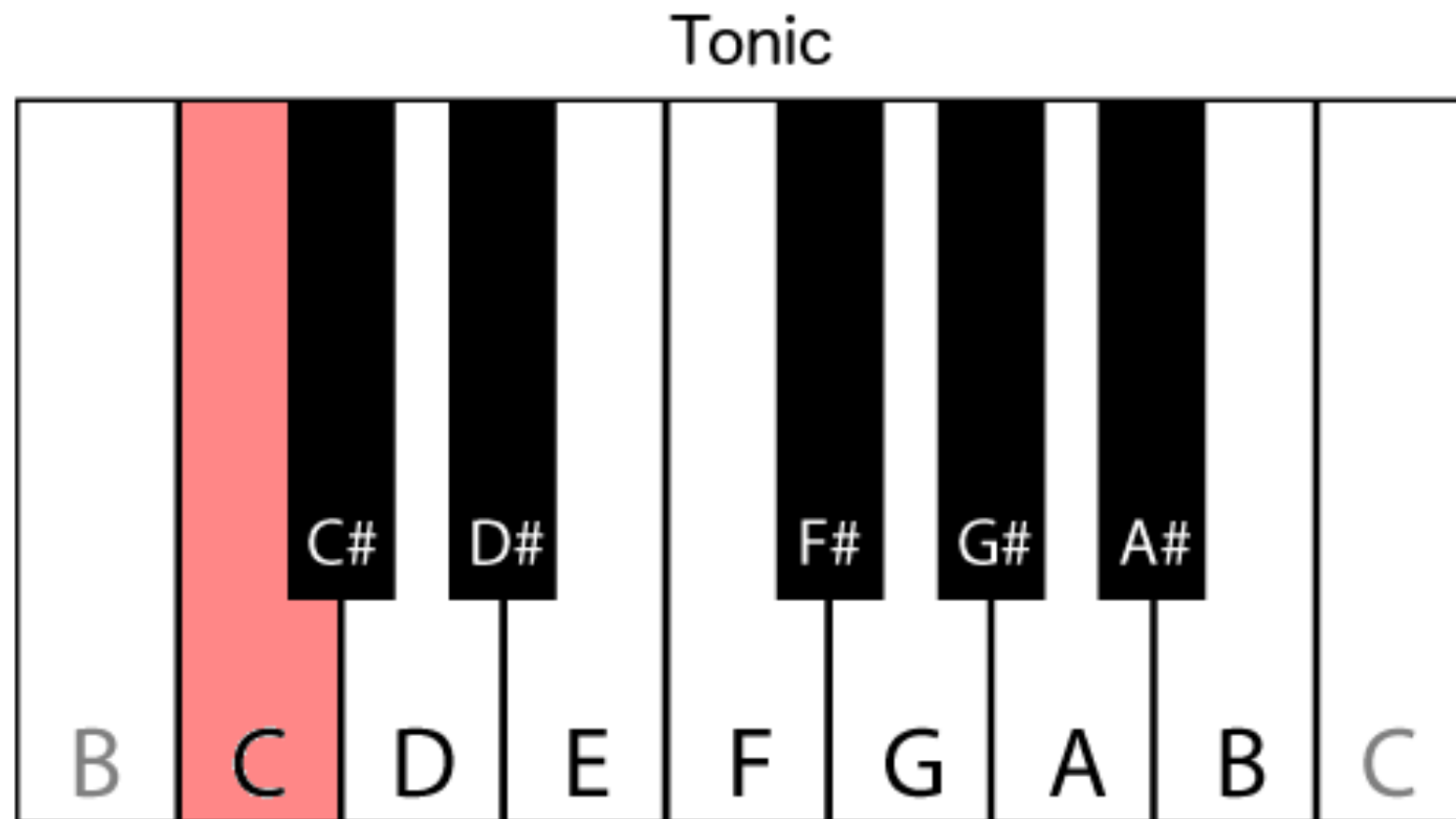
- ▶ A scale is a pattern of notes, usually within an octave
- ▶ In Western music we use the **diatonic scale**



- ▶ This scale contains **seven** distinct pitch classes and is part of a general class of scales known as *heptatonic*
- ▶ Doubling the frequency of a tone in this scale requires going up by 8 notes: hence the term “octave”

Basic Terminology

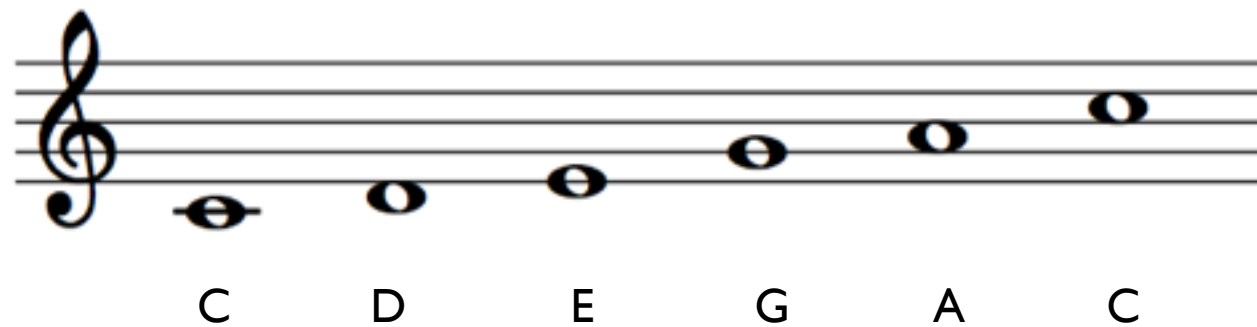
- ▶ Note combinations are described with respect to their position in the scale



- ▶ Ignore the modifiers “major” and “perfect” for now; we’ll come back to those in a few minutes

Pentatonic Scale

- ▶ The diatonic scale (and other heptatonic scales) are found all over the world, but are not universal
- ▶ Traditional Asian music is based on the **pentatonic scale**:



- ▶ Familiar example of the pentatonic scale: opening of “Oh! Susanna” by Stephen Collins Foster



Pentatonic Scale

Arirang (아리랑)

Korean Folk Song

아 리 랑, 아 리 랑, 아 라 리 요... 아 리 랑
A - ri rang, a - ri rang, a - ra__ ri__ yo... A - ri rang
A - ri rang, a - ri rang, a - ra__ ri__ yo... Cross - ing ov -

6
고 개 로 넘 어 간 다. 나 를 버 리 고 가 시 는 님
go - gae - ro__ neo - meo-gan - da. Na - reul beo - ri - go ga - shi-neun nim -
er A - ri - rang moun-tain pass. Dear lov - ed one who a - ban-doned me

12
은 십 리 도 못 가 서 발 병 난 다.
eun__ Shim - ri - do__ mot - ga - seo__ bal - byeong-nan - da.
here shall not walk ten miles be - fore their feet hurt.

Origins of the Scale

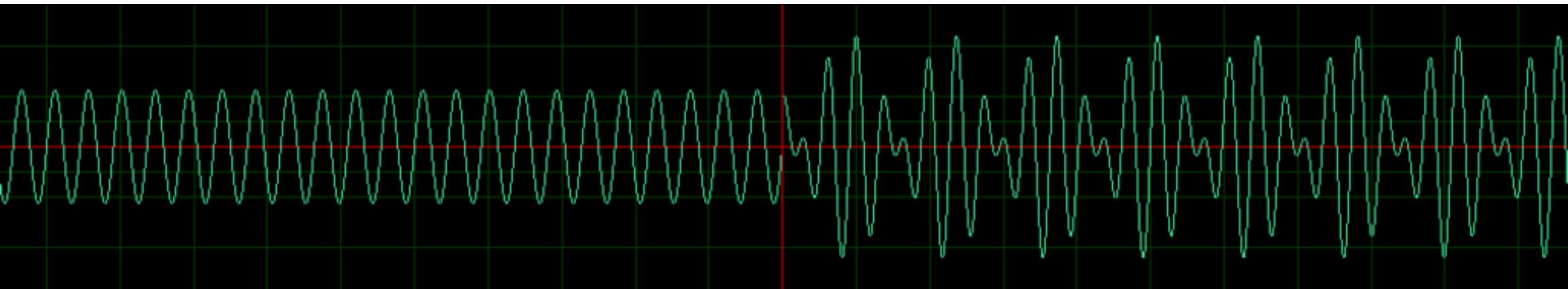
- ▶ Where do these note patterns come from?
- ▶ Why can they be found around the world?



Neolithic bone flutes (7000 BC), Jiahu, China

Psychology of Hearing

- ▶ Our ears interpret musical intervals in terms of ratios
- ▶ **Perfect 4th**: interval between two pitches whose fundamental frequencies form the ratio 4:3
 - Ex: **A4** (440 Hz), **D5** ($586.67 \text{ Hz} = \frac{4}{3} \times 440 \text{ Hz}$)



- Tone sample: 2 s of **A4**, then 2 s of **A4 + D5**

Psychology of Hearing

- ▶ Our ears interpret musical intervals in terms of ratios
- ▶ **Perfect 5th**: interval between two pitches whose fundamental frequencies form the ratio 3:2
 - Ex: **A4** (440 Hz), **E5** ($660 \text{ Hz} = \frac{3}{2} \times 440 \text{ Hz}$)



- Tone sample: 2 s of **A4**, then 2 s of **A4 + E5**

Chromatic Scale

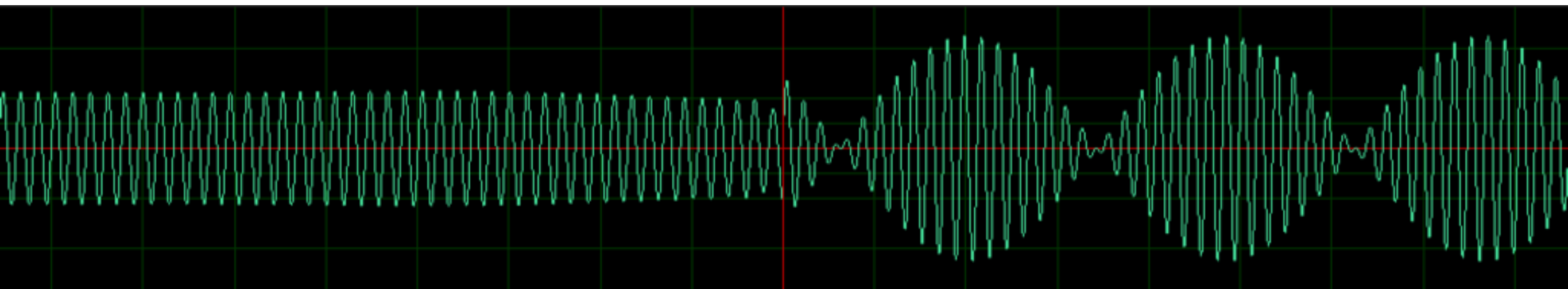
- ▶ The complete scale in an octave is the 12-pitch **chromatic scale**



- ▶ The 12 pitches are separated by half notes, or **semitones**, the smallest musical interval used in Western music. (Q: what about other traditions?)
- ▶ Semitones are quite dissonant when sounded harmonically; i.e., they don't sound pleasant

Dissonant Semitones

- ▶ Why do semitones sound so dissonant?
- ▶ Look at the waveform produced by playing a pure **middle C** (261.63 Hz) and **C#** (277.19 Hz)



- ▶ Significant **beating** is present; our ears don't like it
- ▶ Question: what's the beat frequency?

Origin of Scales and Pitch

- ▶ We like octaves, which are a frequency ratio of 2:1
- ▶ We also like the sound of small integer ratios of frequency, e.g., the perfect 5th (3:2)
- ▶ These kinds of integer ratios tend to show up when you **tune an instrument by ear**, because they “sound right.” Our ears like it when harmonics align due to the lack of dissonant beats
- ▶ However, the manner in which notes are scaled within an octave can be pretty arbitrary
- ▶ Even the pitch of notes has evolved over time

Absolute Pitch

- ▶ In the baroque period, $A = 415$ Hz was the standard



- ▶ Bach used **415 Hz**. Handel used **422.5 Hz**. The international tuning standard of **A4 = 440 Hz** was widely adopted around 1920

Building Scales

Pythagorean Tuning
Just Temperament
Equal Temperament

Overtone series of the String

Berg and Stork, Ch. 3

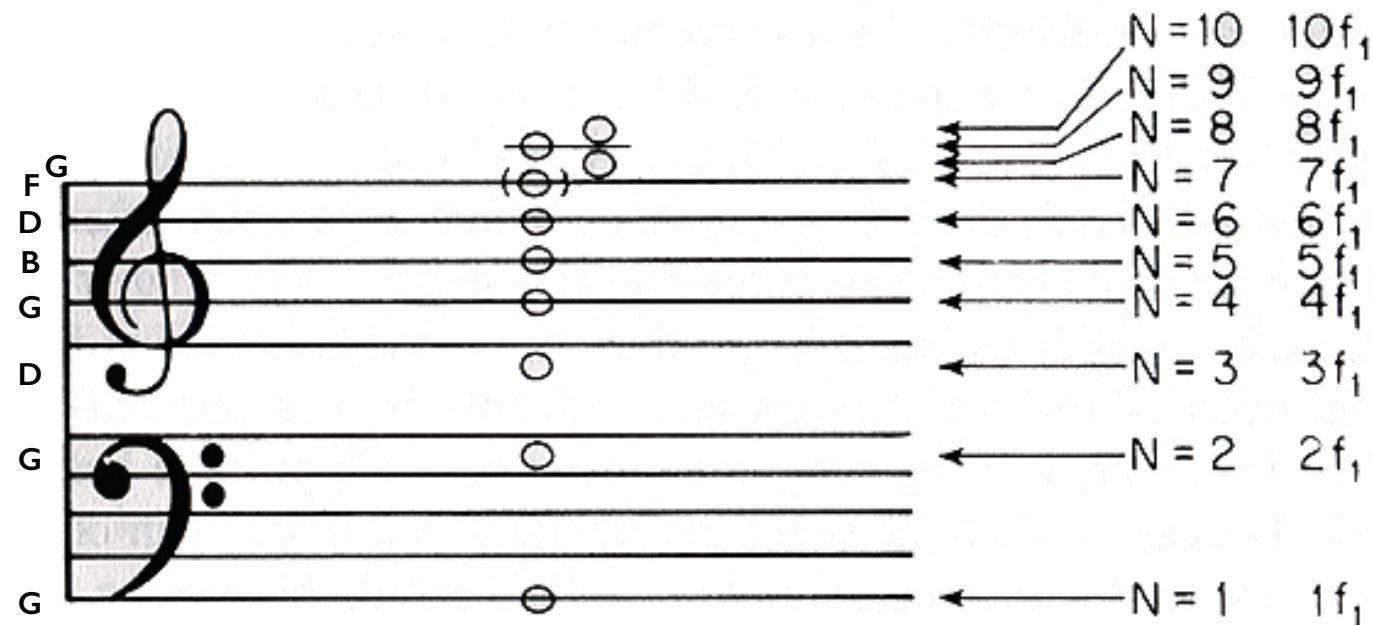


Figure 3-12 Notes on the musical staff having frequencies closest to the notes in the overtone series of G_2 ; each note is labeled by its harmonic number and the frequency of the corresponding harmonic.

TABLE 3-3 MUSICAL INTERVALS BETWEEN THE FUNDAMENTAL AND OTHER NOTES OF THE OVERTONE SERIES

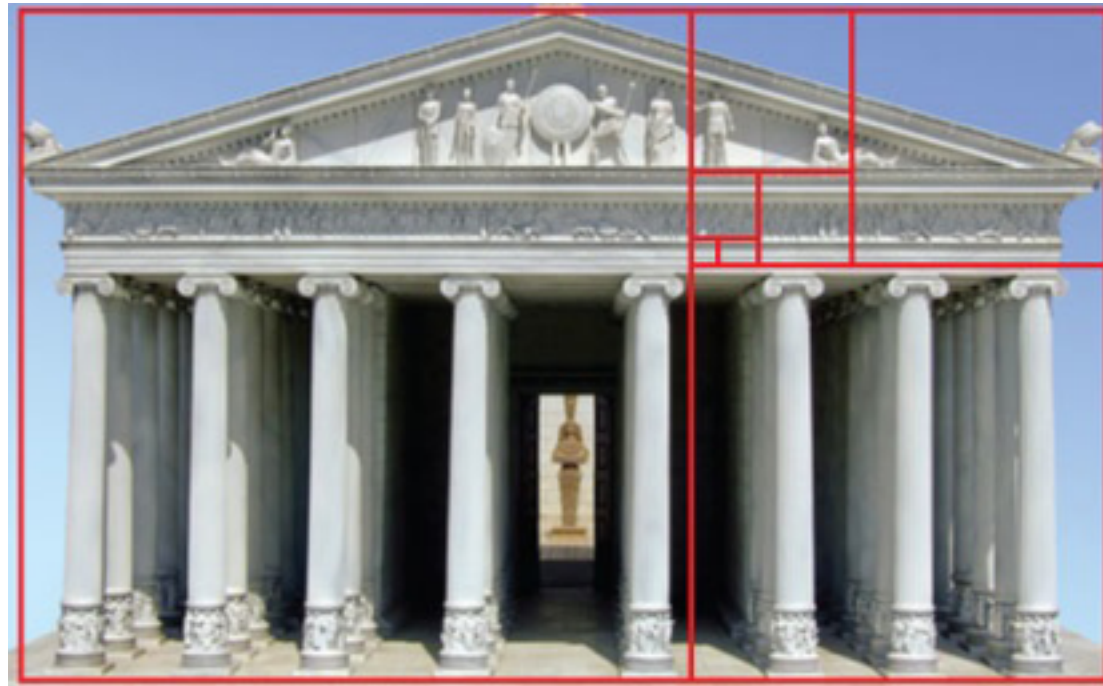
N	f	Interval
1	f_1	Unison
2	$2f_1$	One octave
3	$3f_1$	One octave + one perfect fifth
4	$4f_1$	Two octaves
5	$5f_1$	Two octaves + one major third
6	$6f_1$	Two octaves + one perfect fifth
7	$7f_1$	Two octaves + one minor seventh
8	$8f_1$	Three octaves

Origin of the Diatonic Scale

- ▶ Start with the tonic of the string (f) and go *up* by an octave: you get a doubling of the frequency ($2f$)
- ▶ An octave + 5th gives the string's third harmonic ($3f$)
- ▶ Drop the octave + 5th note *down* by one octave to the first 5th and the frequency is divided by two ($3f/2$)
- ▶ Hence, the perfect fifth is in a $3:2$ ratio with the tonic
- ▶ **Pythagorean tuning**: build up the chromatic scale of 12 notes by climbing up the scale by 5ths and down by octaves

Pythagorean Tuning

- ▶ The Pythagorean tuning system is one of the first theoretical tuning systems in Western music (that we know about)



- ▶ The Pythagorean scale appeals to **symmetry**: you can construct the chromatic scale in terms of simple integer ratios of a fundamental frequency

Pythagorean Temperament

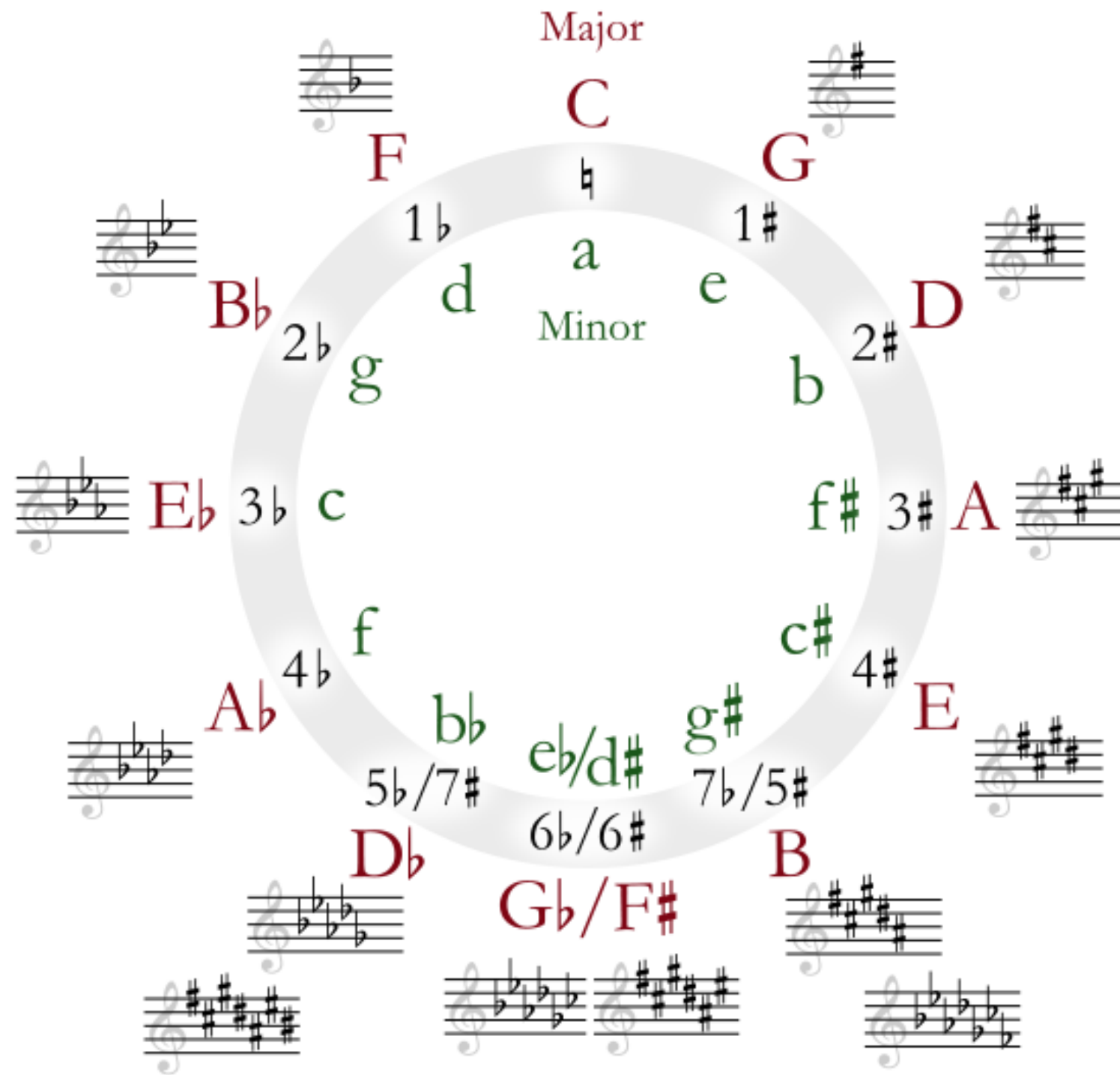
► Go up by **5ths** and down by **8ves** to fill the scale:



Berg and Stork, Ch. 9

<i>Note</i>	<i>Frequency Relation to Tonic</i>	<i>Ratio</i>
C₄	tonic (1.000)	1.0000
G₄	$3/2 \cdot C_4$	1.5000
D₅ → D₄	$1/2 \cdot D_5 = 1/2 \cdot (3/2 \cdot G_4) = 1/2 \cdot (3/2 \cdot (3/2 \cdot C_4)) = 9/8 \cdot C_4$	1.1250
A₄	$3/2 \cdot D_4 = 27/16 \cdot C_4$	1.6875
E₅ → E₄	$1/2 \cdot E_5 = 1/2 \cdot (3/2 \cdot A_4) = 81/64 \cdot C_4$	1.2656
B₄	$3/2 \cdot E_4 = 3/2 \cdot (81/64 \cdot C_4) = 243/128 \cdot C_4$	1.8984
B₃ → F₄#	$3/2 \cdot (1/2 \cdot B_4) = 3/4 \cdot (243/128 \cdot C_4) = 729/512 \cdot C_4$	1.4238
...

Circle of Fifths



- ▶ Another visualization of the construction of the chromatic scale
- ▶ Gives the major and minor keys of the 12 pitches

Circle of 5ths in Composition

- ▶ Popular music example: *Take a Bow* by Muse
- ▶ Starts in the key of D, then goes to G, C, F, etc.

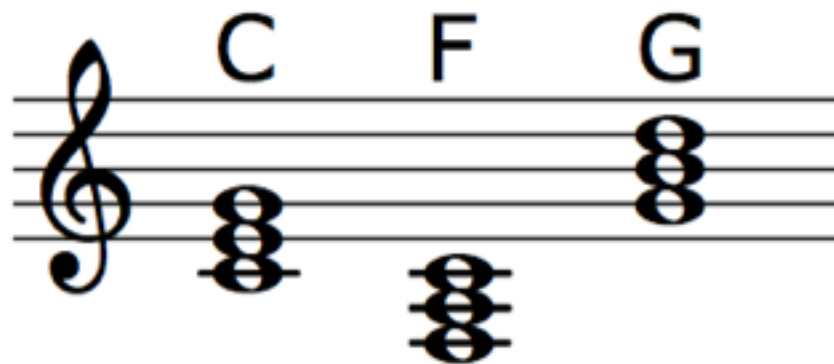


- ▶ Dramatic political song, lyrics are a bit... incendiary. Ignore them, listen for the key changes

Issues with Pythagorean Scale

- ▶ The Pythagorean tuning system is pretty elegant
- ▶ Given just a tonic and the **3:2 perfect 5th** and **2:1 octave ratios**, we can construct the frequencies of all 12 notes in the chromatic scale
- ▶ Unfortunately, the major third (81:64) and minor third (32:27) are pretty **dissonant** in this system

- **Triads?**
- **Chords?**



The Wolf Interval

- ▶ If you go up by a 5th **12 times**, you expect to be **7 octaves** above the starting point
- ▶ But $(3/2)^{12} \approx 129.74$, and $2^7 = 128$; so the circle of fifths doesn't fully close in Pythagorean tuning
- ▶ One of the 5th intervals must not match the prescribed frequency ratio. Therefore, there is a **dissonant beat** (the interval “howls”)



Perfect 5ths and 3rds

- ▶ **Perfect 5th** with frequency ratio 3:2
- ▶ **Wolf 5th** (between C# and A \flat), frequency ratio is $1:2 \frac{1}{3} = 1.4798$
- ▶ **Perfect 3rd** with frequency ratio 5:4
- ▶ **Pythagorean 3rd** with frequency ratio 1.2656
- ▶ Sawtooth waves used to include overtones and make the dissonance in Pythagorean tuning more obvious

Just Temperament

- ▶ A tuning system in which all the frequencies in an octave are related by very simple integer ratios is said to use **just intonation** or **just temperament**
- ▶ Frequency ratios obtained in the most basic form of a **major scale**, relative to the tonic, when using just intonation, are:
 - 1:1, 9:8, 5:4, 4:3, 3:2, 5:3, 15:8, 2:1
- ▶ Can get this by **tuning the 3rds** and allowing some of the 5ths to be slightly out of tune
- ▶ Chords no longer dissonant; richer music is possible

Issues with Just Temperament

- ▶ Unfortunately, simple frequency ratios don't solve the problems of dissonant chords
- ▶ As in the Pythagorean system, there are certain keys and chords that are **unplayable** in just temperament
- ▶ With an integer frequency ratio, **tuning errors** have to accumulate in certain chords
- ▶ **Changing keys** is also tricky; you have to be careful about how frequencies are calculated

Equal Temperament

- ▶ Equal temperament is an attempt to get away from the problem of errors showing up in **certain chords**
- ▶ Idea: tuning errors are distributed equally over all possible triads
- ▶ All triads become equal, making **key changes** much, much easier
- ▶ Cost: mild dissonance is present in many chords

12 Tone Equal Temperament

- ▶ We want to fit 12 tones equally with an octave, i.e., between frequencies f and $2f$
- ▶ How to do it?
- ▶ Could try to space the frequencies evenly, in other words, $f_n = (1 + n/12)f$


Note	C	C#	D	D#	E	F	F#	G	G#	A	B ♭	B	C
n	0	1	2	3	4	5	6	7	8	9	10	11	12
Ratio	1.00	1.083	1.167	1.25	1.333	1.417	1.5	1.583	1.667	1.75	1.833	1.917	2

- ▶ Problem: the diatonic scale sounds *just awful*


12 Tone Equal Temperament

- ▶ Better: use a multiplicative factor such that $f_n = a^{n/12}f$
- ▶ For $f_{12} = 2f$ (one octave) we need $a = 2$. Therefore,


Note	C	C#	D	D#	E	F	F#	G	G#	A	B ♭	B	C
n	0	1	2	3	4	5	6	7	8	9	10	11	12
Ratio	1.00	1.0595	1.122	1.189	1.26	1.335	1.414	1.498	1.587	1.682	1.782	1.888	2



Major 3rd:
 $5/4 = 1.25$



Perfect 4th:
 $4/3 = 1.333$



Perfect 5th:
 $3/2 = 1.5$

- ▶ Diatonic scale sounds pretty good!

Logarithmic Scale

- ▶ The equal-tempered scale is not equally spaced in units of f ; it is **equally spaced in units of $\log f$**
- ▶ Example: observe increase in $\log f$ per semitone

Note	C	C#	D	D#	E	F	F#	G	G#	A	B \flat	B	C
n	0	1	2	3	4	5	6	7	8	9	10	11	12
f_n	1.00	1.0595	1.122	1.189	1.26	1.335	1.414	1.498	1.587	1.682	1.782	1.888	2
$\log f_n$	0.000	0.025	0.05	0.075	0.1	0.125	0.151	0.176	0.201	0.226	0.251	0.276	0.301
$\log f_n/f_{n-1}$		0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025	0.025

Logarithmic Scale



- ▶ The logarithmic scale shows up in instrument design
- ▶ Notice how the guitar frets get closer together as you move down the neck
- ▶ They are equally spaced by the same multiplicative factor
 $2^{1/12} \approx 1.0595$
- ▶ Equal temperament makes design easy, as long as you remember this factor

Musical Cents

- ▶ Cents are a subdivision of the semitone that you will use for tuning your instruments
- ▶ Intervals between notes are described using cents
- ▶ The definition is: $\text{¢} = 1200 \log_2(f_2 / f_1)$
- ▶ Alternatively: $f_2 / f_1 = 2^{\text{¢}/1200}$
- ▶ If f_2 is one octave higher than f_1 , then $f_2 = 2f_1$, and therefore $\text{¢} = 1200$
- ▶ I.e., there are 1200 cents per octave, and 100 cents per semitone

Intervals in Base-10

- ▶ If you don't like working in base-2 logarithms, you can convert to base-10
- ▶ Note: $1200 \cdot \log_2(f_2/f_1) \cong 3986 \cdot \log(f_2/f_1)$
- ▶ Alternatively, $f_2 = f_1 \cdot 2^{n/1200} \cong f_1 \cdot 10^{n/3986}$

Notes + Cents \rightarrow Frequency

► How to use the digital tuner: what is $C4\# + 25\text{¢}$?

- $C4\# = 277.18 \text{ Hz}$

- $2^{25/1200} = 1.0145453$

- $C4\# + 25\text{¢} = 2^{25/1200} \times 277.18 \text{ Hz} = 281.21 \text{ Hz}$

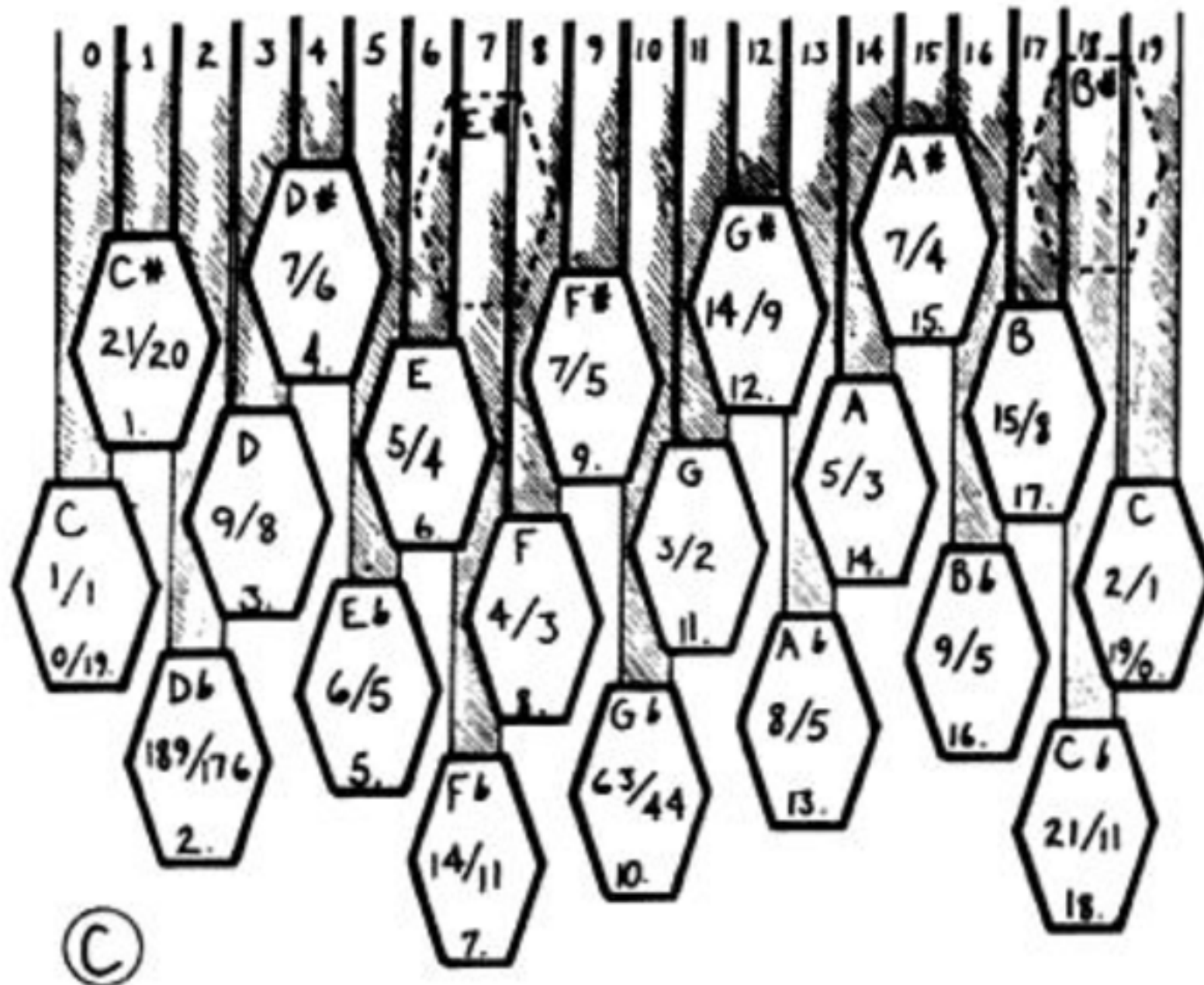
► What is $C4\# - 25\text{¢}$?

Other Equal-Tempered Scales

- ▶ 12-tone equal temperament is special in that it is the **smallest division** of the octave that does a reasonable job of **approximating** the just intervals we like to hear
- ▶ But it's not the only scale that makes a good approximation. Others include:
 - 19-tone scale
 - 24-tone scale (a.k.a. **quarter-tone scale**)
 - 31-tone scale
 - 53-tone scale

19-Tone Keyboard Layouts

- Proposed layout for 19-tone keyboards (from Hopkin, Ch. 3)



Summary

- ▶ The intervals common to music around the world (octaves, 5ths, 3rds, ...) are based on simple integer ratios of frequencies
- ▶ We like these ratios because they are consonant; we dislike high-integer ratios because they sound dissonant, due to beats
- ▶ **Just intonation**: scale you get when tuning by ear
- ▶ **Equal temperament**: equally spaced semitones on a logarithmic scale
- ▶ Reading: Hopkin Ch. 3, Berg and Stork Ch. 9