PHY473a: Course project

Generating the solar magnetic field using spherical harmonics

by

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April 21, 2017

Contents

Contents i

List of Figures i

1 Aim 1

2 Theoretical framework 1
- 2.1 Toroidal and polar magnetic fields . . . 1
- 2.2 Equations and method used in program . . . 2

3 Results 3

4 Conclusions and Future Prospects 9

5 Acknowledgements 9

References 9

List of Figures

- 3.1 Y_1^0 field-vectors . . . 3
- 3.2 Y_2^0 field-vectors . . . 3
- 3.3 Y_3^0 field-vectors . . . 4
- 3.4 Dipole and octapole flow lines . . . 4
- 3.5 $\mathbf{B}_P$ for all spherical harmonics . . . 5
- 3.6 $\mathbf{B}_T$ for all spherical harmonics . . . 6
- 3.7 An initial guess that gave spots . . . 7
- 3.8 Better approximation for sun spots . . . 7
- 3.9 The final result . . . 8

Aim

The main objectives of the project are:

- To express potentials associated with polar and toroidal components of magnetic field of sun as spherical harmonics.
- To study magnetic field line structure due to each spherical harmonic component.
- To verify pole structure generated by higher harmonics.
- To determine coefficients of harmonics needed to generate sun spots.
- To ultimately try and generate a close representation of the observed magnetic field of the sun.

Theoretical framework

2.1 Toroidal and polar magnetic fields

The solar magnetic field can be decomposed into toroidal ($\mathbf{B}_T$) and polar ($\mathbf{B}_P$) components (see [1]). The components are given by:

$$\mathbf{B}_T = \nabla \times (T\mathbf{r}) \qquad \mathbf{B}_P = \nabla \times \nabla \times (S\mathbf{r}) \tag{2.1}$$

Where T and S are scalar fields. In our project we assume T $\to$ 0 and S $\to$ 0 as distance from the center r $\to \infty$. Besides, we assume that T and S can be written as (both are like potentials)

$$T(\mathbf{r}) = \sum_i C_i \frac{1}{|\mathbf{r} - \mathbf{a_i}|} \qquad S(\mathbf{r}) = \sum_i C'_i \frac{1}{|\mathbf{r} - \mathbf{b_i}|} \tag{2.2}$$

On doing multipole expansion (Laplace expansion) of $\frac{1}{|\mathbf{r}-\mathbf{a_i}|}$, we get

$$\frac{1}{|\mathbf{r} - \mathbf{a_i}|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} \frac{Y_l^m(\theta, \phi)}{r^{l+1}} \tag{2.3}$$

Where α's are constants. Hence this expansion gives us:

$$T(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \tau_{lm} \frac{Y_l^m(\theta, \phi)}{r^{l+1}} \qquad S(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} \frac{Y_l^m(\theta, \phi)}{r^{l+1}} \tag{2.4}$$

In our project we first show what is the field structure due to each individual component. Now for $l = 0$, we get $\frac{Y_l^m(\theta,\phi)}{r^{l+1}}\mathbf{r} = \mathit{constant} \times \hat{\mathbf{r}}$. Which has a curl zero. Thus $l = 0$ does not contribute at all in generating magnetic fields. So, we can write (2.4) as:

$$T(\mathbf{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\tau_{lm}\frac{Y_l^m(\theta,\phi)}{r^{l+1}} \qquad\qquad S(\mathbf{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sigma_{lm}\frac{Y_l^m(\theta,\phi)}{r^{l+1}} \tag{2.5}$$

2.2 Equations and method used in program

In our program, we symbolically calculate $\mathbf{B}_T$ and $\mathbf{B}_P$ by calculating the curls of the spherical harmonics. Only the first three harmonics were used for ease of calculation.

$Y_l^m(\theta,\phi)$ are calculated recursively using the formulation:

$$Y_l^{m-1}(\theta,\phi) = \frac{1}{\sqrt{(l+m)(l-m+1)}}e^{-i\phi}\Big(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Big)Y_l^m(\theta,\phi) \tag{2.6}$$

Now, $Y_l^l(\theta,\phi)$ is given by:

$$Y_l^l(\theta,\phi) = \left[\frac{(-1)^l}{2^l l!}\right]\sqrt{\frac{[(2l+1)(2l)!]}{4\pi}}e^{il\phi}\sin^l\theta \tag{2.7}$$

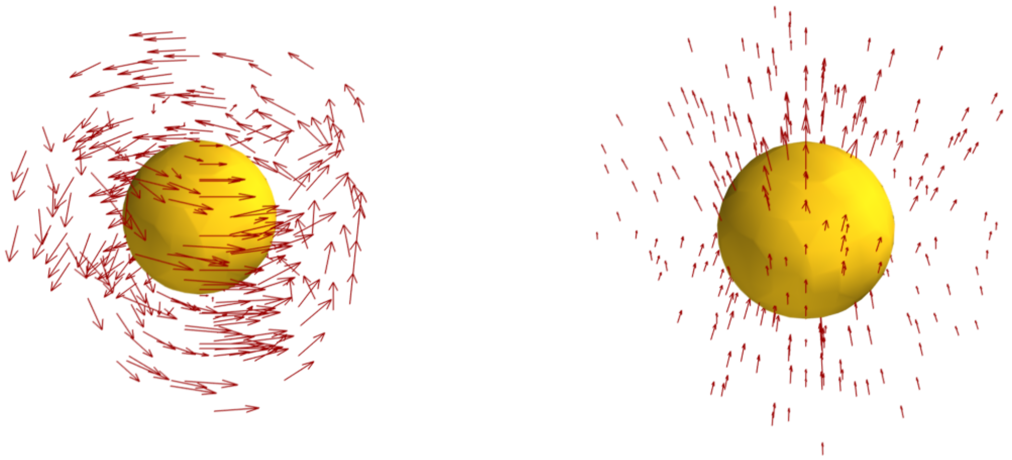
The curl was then calculated in radial polar coordinates using (for a vector $\mathbf{F}$):

$$\begin{aligned}
(\nabla\times\mathbf{F})_{\hat{\mathbf{r}}} &= \frac{1}{r\sin\theta}\Big[\partial_\theta(F_\phi\sin\theta) - \partial_\phi(F_\theta)\Big] \\
(\nabla\times\mathbf{F})_{\hat{\theta}} &= \frac{1}{r}\Big[\frac{1}{\sin\theta}\partial_\phi F_r - \partial_r(rF_\phi)\Big] \\
(\nabla\times\mathbf{F})_{\hat{\phi}} &= \frac{1}{r}\Big[\partial_r(rF_\theta) - \partial_\theta F_r\Big]
\end{aligned} \tag{2.8}$$

All calculations were done using symbolic python, for accuracy. The plots were then generated using Mayavi. To view the behaviour of the field well, without multiple confusing field lines, we plotted an effective map to a 2D surface (sphere). For this, the magnitude of vector field at any position was multiplied by sign of radial component of magnetic field (in plots where only $\mathbf{B}_T$ is shown, sign of ϕ component was used to represent the toroidal field well) and then plotted on the surface of sun.

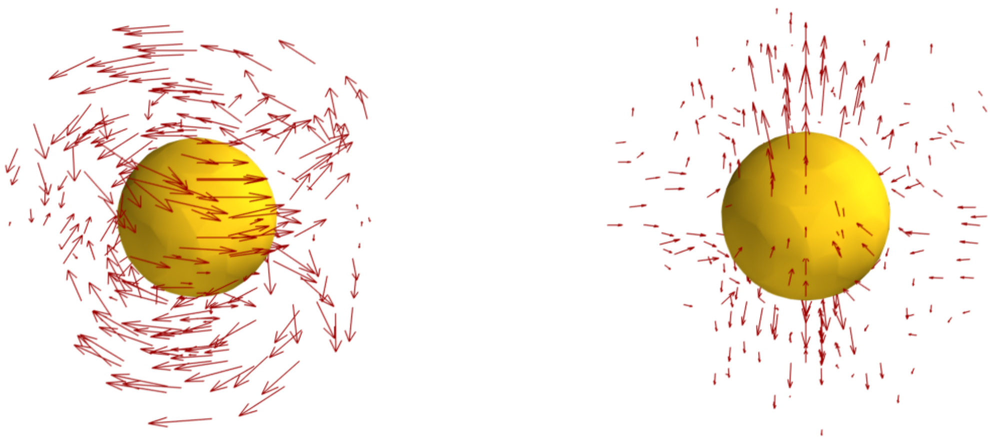
Results

- Verification of method: Plots for dipole (Y_1^0), quadrupole (Y_2^0) and octapole (Y_3^0)



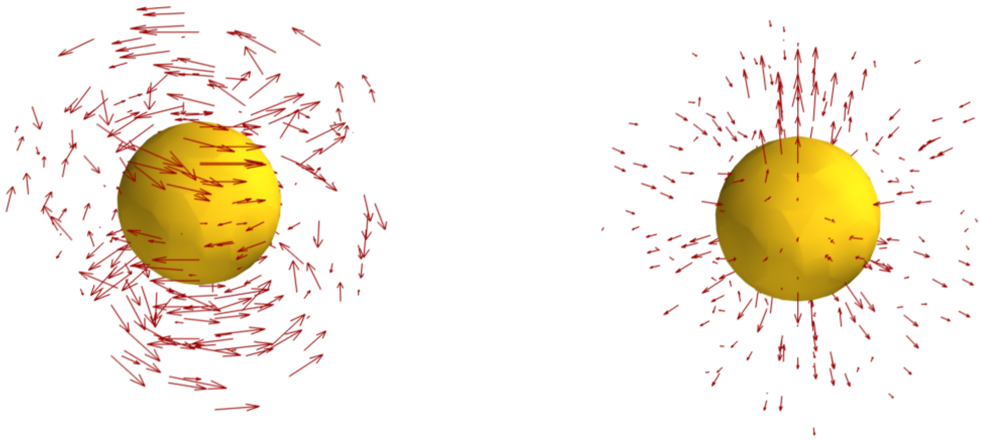
(a) toroidal Y_1^0 (b) polar Y_1^0

Figure 3.1: Y_1^0 field-vectors



(a) toroidal Y_2^0 (b) polar Y_2^0

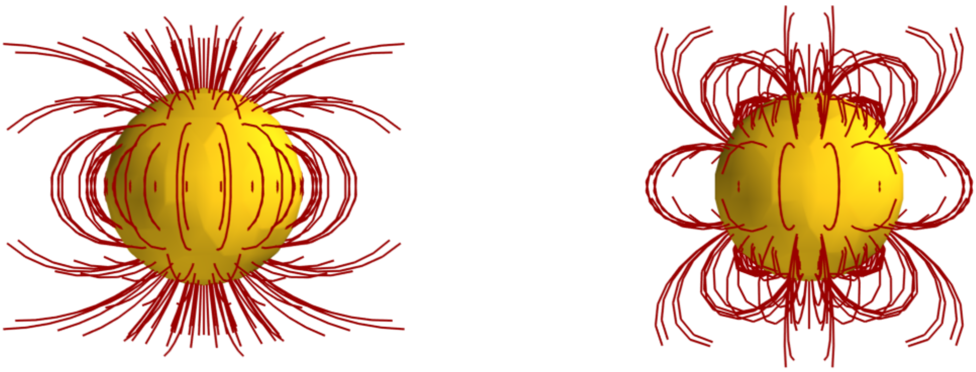
Figure 3.2: Y_2^0 field-vectors



(a) toroidal Y_3^0

(b) polar Y_3^0

Figure 3.3: Y_3^0 field-vectors



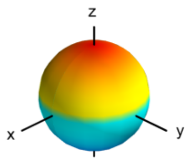
(a) Polar Y_1^0

(b) Polar Y_3^0

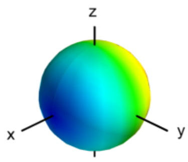
Figure 3.4: Dipole and octapole flow lines

As we can see, the all three of the Y_l^0 plot sets give the required results. Also, combining the dipole and octapole fields seems to suggest we could obtain small sun spots, as the larger components could effectively cancel each other out.

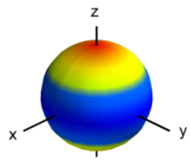
- Plots for individual spherical harmonics Y_l^m *(showing only positive m's due to symmetry)*



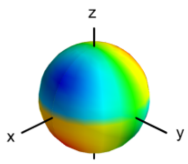
(a) Y_1^0



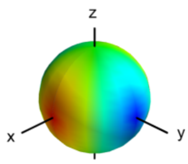
(b) Y_1^1



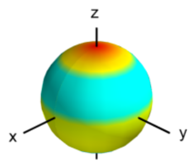
(c) Y_2^0



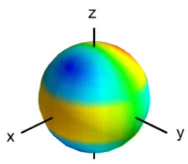
(d) Y_2^1



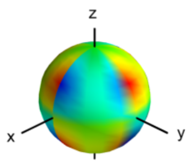
(e) Y_2^2



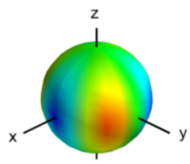
(f) Y_3^0



(g) Y_3^1

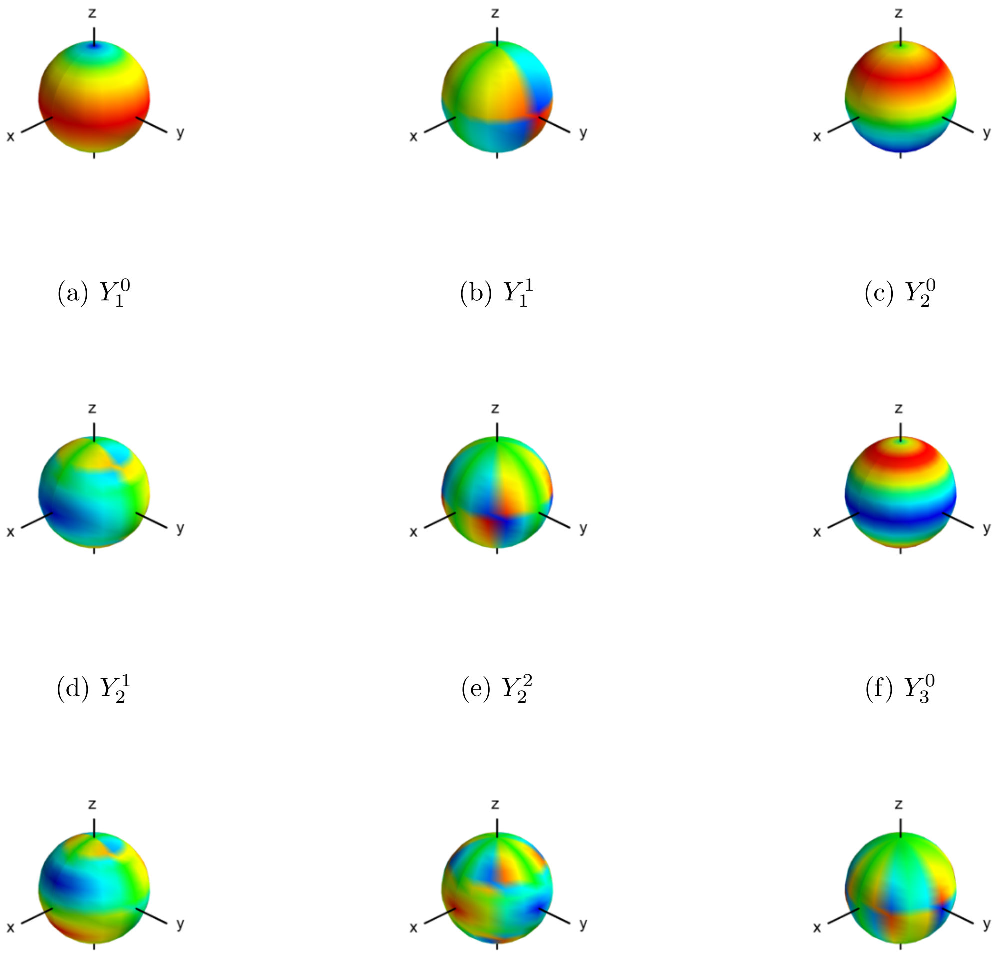


(h) Y_3^2



(i) Y_3^3

Figure 3.5: $\mathbf{B}_P$ for all spherical harmonics



(a) Y_1^0 (b) Y_1^1 (c) Y_2^0

(d) Y_2^1 (e) Y_2^2 (f) Y_3^0

(g) Y_3^1 (h) Y_3^2 (i) Y_3^3

Figure 3.6: $\mathbf{B}_T$ for all spherical harmonics

Note that in the case of $\mathbf{B}_T$, $T\mathbf{r}$ only has the radial component. Hence by (2.1), the (scalar) r dependence of T does not change anything in the plot.

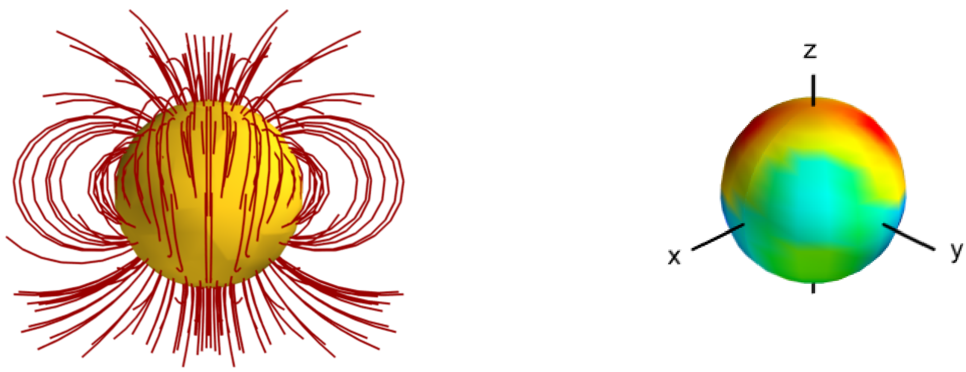
Also, note how the toroidal ($\mathbf{B}_T$) and polar ($\mathbf{B}_P$) fields of Y_3^2 and Y_2^2 seem to give pairs of blue (negative, inward field lines) and red/yellow (positive, outward field lines) spots. This is exactly how sun spots would appear in such a plot representation. Hence this led us to believe that we could recreate the observed sun spot structures using combinations of these harmonics.

- The final plots using appropriate coefficients:

(a) Field-lines (b) Spot-structure

Figure 3.7: An initial guess that gave spots

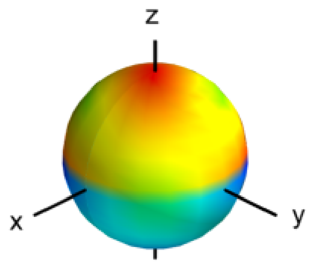
In our initial guess, we randomly played around with coefficients to see if we could generate spots. We got our result, but realised we had to take into account the imaginary components of the harmonics too (using complex coefficients).



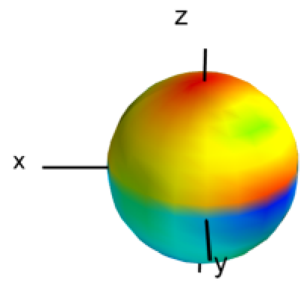
(a) Field-lines (b) Spot-structure

Figure 3.8: Better approximation for sun spots

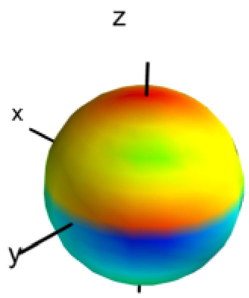
We could also see that Y_3^2 seemed to be giving interesting spots that we could use. Hence in our next (better) try, we simulated spots using this component, while providing the natural magnetic field through Y_1^0 and Y_3^0.



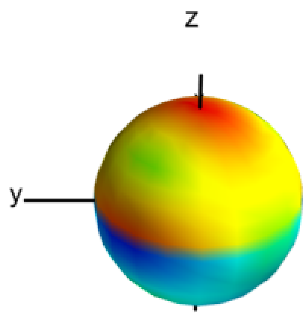
(a)



(b)



(c)



(d)

Figure 3.9: The final result

After varying the coefficients further, we came up with a plot that gave reasonable sun spots. As we can see, the spots appear near the equator as small bands, while the pole structure remains intact.

The coefficients used for this final plot were:

field	Y_1^{-1}	Y_1^0	Y_1^1	Y_2^{-2}	Y_2^{-1}	Y_2^0	Y_2^1	Y_2^2
$\mathbf{B}_T$	0	0	0	0	-2.6 * (1 - ι)	3	2.6 * (1 + ι)	0
$\mathbf{B}_P$	0	1.8	0	0	0	0	0	0

field	Y_3^{-3}	Y_3^{-2}	Y_3^{-1}	Y_3^0	Y_3^1	Y_3^2	Y_3^3
$\mathbf{B}_T$	0	0	1	1.1	-1	0	0
$\mathbf{B}_P$	0	-1	-1.2	1.5	1.2	1	0

Conclusions and Future Prospects

We conclude that the multipole expansion of the toroidal and polar fields indeed does seem to generate a close approximation to the sun's field. Surprisingly, with components up to just the third harmonic, we can already produce field lines that strongly resemble the structure of the sun's spots. We believe hence that our project was successful, and can be used in the future to generate the magnetic field of any object, given the coefficients.

This project can be extended in multiple ways. The primary aim of the project was of course, to realise the sun's magnetic field in terms of the spherical harmonics. Currently, our combined plot is generated by randomly guessed coefficients for the components. However if data of sun's magnetic field is available, then one can calculate these coefficients more precisely. This can help us understand the nature and reason of the spot structure and it's change in the sun. We can also try going to higher harmonics to more accurately represent the phenomena we see in the sun. Generating these coefficients as a function of time will even allow us to possibly view phenomena like the pole reversal or solar storms.

Acknowledgements

We sincerely express gratitude to our mentor Dr. Mahendra K. Verma for giving us this valuable opportunity to learn under him and suggesting such a unique idea for a project. He has provided immeasurable amount of support and guidance throughout the project, along with material that gave us numerous details. We would also like to the TA's of the course. Their suggestions have helped us a lot.

References

[1] Mathematical Aspects of Natural Dynamos. Edited by E. Dormy and A.M. Soward. Grenoble Sciences.

[2] Wikipedia article on multipole expansion: `https://en.wikipedia.org/wiki/Multipole_expansion`

[3] Mayavi documentation: `http://docs.enthought.com/mayavi/mayavi/`

[4] Python documentation: `https://docs.python.org/2.7/`