Homework 1 – February 1, 2006 **Solution** prepared by Tobin Fricke
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1. Moments of the Poisson distribution. The mean of a discrete random variable ξ is calculated as $\langle \xi \rangle = \sum kP[\xi = k]$ where the sum is over all possible values k the random variable may attain. For a random variable ξ with Poisson distribution with parameter λ, the probability mass function is $P[\xi = k] = (\lambda^k/k!)\exp(k)$.

$$\langle \xi \rangle = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \lambda \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} = e^{-\lambda} e^{\lambda} \lambda = \boxed{\lambda}$$

Instead of directly calculating $\sigma^2 = \left\langle (\xi - \langle \xi \rangle)^2 \right\rangle = \langle \xi^2 \rangle - \langle \xi \rangle^2$ or even $\langle \xi^2 \rangle$, it is easiest to first compute $\langle \xi(\xi-1) \rangle = \langle \xi^2 \rangle - \langle \xi \rangle$, following a trick similar to the above:

$$\langle \xi(\xi-1) \rangle = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda} \lambda^2 \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} = e^{-\lambda} e^{\lambda} \lambda^2 = \lambda^2$$

Using $\langle \xi(\xi-1) \rangle = \lambda^2$ and $\langle \xi \rangle = \lambda$ we can find the variance σ^2:

$$\sigma^2 = \langle \xi^2 \rangle - \langle \xi \rangle^2 = \langle \xi(\xi-1) \rangle + \langle \xi \rangle - \langle \xi \rangle^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \implies \boxed{\sigma = \sqrt{\lambda}}$$

The same trick works for the computation of $\langle \xi^3 \rangle$ (the "third moment" of the distribution):

$$\langle \xi(\xi-1)(\xi-2) \rangle = \sum_{k=0}^{\infty} k(k-1)(k-2) \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \lambda^3 \sum_{k=2}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = e^{-\lambda} \lambda^3 \sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!} = \lambda^3$$

$$\left\langle \xi^3 \right\rangle = \langle \xi(\xi-1)(\xi-2) \rangle + 3\langle \xi^2 \rangle - 2\langle \xi \rangle = \lambda^3 + 3\left(\sigma^2 + \langle \xi \rangle^2\right) - 2\langle \xi \rangle = \lambda^3 + 3\left(\lambda + \lambda^2\right) - 2\lambda = \boxed{\lambda^3 + 3\lambda^2 + \lambda}$$

Aside: We will see later that, for a random variable ξ, we can define a function $f_\xi(t)$ called ξ's *characteristic function* that is the Fourier transform of its probability distribution, $f_\xi(t) = \langle \exp\{it\xi\} \rangle$. Given this function, it turns out that we may easily compute the nth moment $\langle \xi^n \rangle$ of ξ as $\langle \xi^n \rangle = (1/i^n)(\partial^n/\partial t^n) f_\xi(t)|_{t=0}$. For a random variable ξ with Poisson distribution, one can find $f_\xi(t) = \exp\{\lambda(\exp(it) - 1)\}$. The following Mathematica code computes the characteristic function and uses it to generate a table of moments:

```
p[k_] := Exp[-\[Lambda]] \[Lambda]^k / Factorial[k]

f[t_] = Sum[Exp[I k t] p[k], {k, 0, Infinity}]

Table[{\[LeftAngleBracket]\[Xi]^n\[RightAngleBracket],
      I^(-n) D[f[t], {t, n}] /. t -> 0 // Expand}, {n, 1, 4}] // TableForm
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2. Convergence of the binomial distribution to the Poisson distribution. For a random variable ξ with Binomial distribution, given N trials and probabilities p and $q = (1-p)$ for success and failure, the probability of exactly k successes is:

$$P[\xi = k] = \binom{N}{k} p^k q^{N-k}$$

$$P[\xi = k] = \frac{N!}{(N-k)!k!} p^k (1-p)^{N-k}$$

$$P[\xi = k] = \frac{N(N-1)(N-2)\cdots(N-k+1)}{k!} p^k (1-p)^{N-k}$$

Notice that there are k terms in the denominator of the leading fraction. Divide the top and bottom by N^k:

$$P[\xi = k] = N^k \frac{N}{N} \frac{N-1}{N} \frac{N-2}{N} \cdots \frac{N-k+1}{N} \frac{1}{k!} p^k (1-p)^{N-k}$$

$$P[\xi = k] = \frac{N^k p^k}{k!} \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \cdots \left(1 - \frac{k-1}{N}\right) (1-p)^{N-k}$$

We write p in terms of the number of expected successes, $p = \lambda/N$:

$$P[\xi = k] = \frac{\lambda^k}{k!} \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \cdots \left(1 - \frac{k-1}{N}\right) \left(1 - \frac{\lambda}{N}\right)^{N-k}$$

To show convergence to the Poisson distribution, we take the limit $N \to \infty$. We'll need the identity $\lim_{N\to\infty} (1 - \lambda/N)^N \to \exp(-\lambda)$, which we may derive:

$$\lim_{N\to\infty} \left(1 + \frac{\lambda}{N}\right)^N = \lim_{N\to\infty} \sum_{k=0}^{N} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k = \lim_{N\to\infty} \sum_{k=0}^{N} \left(\prod_{i=0}^{k-1} \frac{N-i}{N}\right) \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda$$

Our expression for $P[\xi = k]$ was:

$$P[\xi = k] = \frac{\lambda^k e^{-\lambda}}{k!} \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \cdots \left(1 - \frac{k-1}{N}\right) \left(1 - \frac{\lambda}{N}\right)^{N-k}$$

In the limit $N \to \infty$ all of the $(1 - \cdot/N)$ terms will go to unity and the final term will go to $\exp(-\lambda)$ as we just saw, and we obtain the familiar Poisson distribution:

$$\lim_{N\to\infty, Np\to\lambda} P[\xi = k] = \frac{\lambda^k e^{-\lambda}}{k!}$$

We are, however, interested in quantifying the error when we use the Poisson distribution to approximate a Binomial distribution with "large" N. First we must also expand the $(1 - \lambda/N)^N$ term in powers of N:

$$\left(1 - \frac{\lambda}{N}\right)^{N-k} = \exp\left\{(N-k)\log\left[1 - \frac{\lambda}{N}\right]\right\}$$

Expand $\log(1-x)$ in a power series about $x_0 = 1$:

$$\log(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

Using that expansion:

$$\left(1-\frac{\lambda}{N}\right)^{N-k} = \exp\left\{-(N-k)\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{\lambda}{N}\right)^{j}\right\}$$

Working out the term in curly braces:

$$-(N-k)\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{\lambda}{N}\right)^{j} = -\sum_{j'=0}^{\infty}\frac{1}{j'+1}\frac{\lambda^{j'+1}}{N^{j'}} + \sum_{j=1}^{\infty}\frac{k}{j}\frac{\lambda^{j}}{N^{j}} = -\lambda - \sum_{j=1}^{\infty}\left(\frac{\lambda}{N}\right)^{j}\left(\frac{\lambda}{j+1}-\frac{k}{j}\right)$$

$$= -\lambda + \frac{k\,\lambda}{N} + \frac{k\,\lambda^2}{2\,N^2} - \frac{\lambda^2}{2\,N} - \frac{\lambda^3}{3\,N^2} + \cdots$$

$$\left(1-\frac{\lambda}{N}\right)^{N-k} = \exp(-\lambda)\exp\left\{\frac{k\,\lambda}{N} + \frac{k\,\lambda^2}{2\,N^2} - \frac{\lambda^2}{2\,N} - \frac{\lambda^3}{3\,N^2} + \cdots\right\}$$

Putting everything together, we have:

$$P[\xi = k] = \frac{\lambda^k e^{-\lambda}}{k!}\left(\prod_{i=0}^{k-1}\left(1-\frac{i}{N}\right)\right)\exp\left\{\frac{k\,\lambda}{N} + \frac{k\,\lambda^2}{2\,N^2} - \frac{\lambda^2}{2\,N} - \frac{\lambda^3}{3\,N^2} + \cdots\right\}$$

We can expand the exponential, too, as a power series about 0, and we may group the series by powers of $1/N$. Expanding the product in the middle expression and multiplying by this power series will result in another series in powers of $1/N$. To first order, we will see that the estimate is correct. The next term, in $1/N$ is the first error term. Therefore the error of our estimate is of order $1/N$.

3. The Stirling Approximation. The Stirling approximation (in one form) is:

$$\boxed{n! \approx n^n e^{-n} \sqrt{n} \sqrt{2\pi}}$$

This bit of Mathematica produces a decent table:

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stirling[n_] := n^n Exp[-n] Sqrt[n] Sqrt[2 Pi]

Table[{n, N[Factorial[n]], N[stirling[n]],
  N[stirling[n]/Factorial[n] - 1]}, {n, 1, 20}] // TableForm
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n	$n!$	$n^n e^{-n} \sqrt{n} \sqrt{2\pi}$	relative error
1	1.	0.922137	-0.077863
2	2.	1.919	-0.0404978
3	6.	5.83621	-0.0272984
4	24.	23.5062	-0.020576
5	120.	118.019	-0.0165069
6	720.	710.078	-0.0137803
7	5040.	4980.4	-0.0118262
8	40320.	39902.4	-0.0103573
9	362880.	359537.	-0.00921276
10	3.6288×10^{6}	3.5987×10^{6}	-0.00829596
11	3.99168×10^{7}	3.96156×10^{7}	-0.00754507
12	4.79002×10^{8}	4.75687×10^{8}	-0.00691879
13	6.22702×10^{9}	6.18724×10^{9}	-0.0063885
14	8.71783×10^{10}	8.6661×10^{10}	-0.0059337
15	1.30767×10^{12}	1.30043×10^{12}	-0.00553933
16	2.09228×10^{13}	2.08141×10^{13}	-0.00519412
17	3.55687×10^{14}	3.53948×10^{14}	-0.0048894
18	6.40237×10^{15}	6.3728×10^{15}	-0.00461846
19	1.21645×10^{17}	1.21113×10^{17}	-0.00437596
20	2.4329×10^{18}	2.42279×10^{18}	-0.00415765

This form of the Stirling approximation is very accurate. Its accuracy is better than 10% for $n \geq 1$ and better than 1% for $n \geq 9$. The worst error, at $n = 1$ is about 7.8%. In this form, the approximation is indeterminate for $n = 0$, since the approximation involves an n^n term.

In thermodynamics and statistical mechanics, it is common to use a 'Stirling' approximation for $\log(n!)$:

$$\boxed{\log(n!) \approx (n \log n) - n}$$

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