7.1. Waiting times in a Poisson Process A Geiger counter emits a click each time a radioactive decay happens. If the average number of decays in unit time is $\lambda$, what is the probability distribution of the time interval between clicks?

The decay of a bulk quantity of radioactive material is one example of a Poisson process. For a Poisson process, we expect the number of events occurring in a unit time interval to follow the Poisson distribution,

$$p[k \text{ events in unit time}] = e^{-\lambda} \lambda^k / k!,$$

where $\lambda$ is the expected number of events in the unit time interval. This must hold for any notion of a “unit time interval,” so we may interpret $\lambda$ as the rate of events. The distribution of number of events occurring in an arbitrary time duration $t$ is therefore Poisson, with parameter $\lambda t$.

The probability of receiving exactly $k$ events in a time interval $t$ is therefore

$$p[N_t = k] = e^{-\lambda t} (\lambda t)^k / k!.$$ 

The condition of having a waiting time $T$ until the first event is equivalent to having exactly zero events in time $N_T$ followed by exactly one event in the following time $dt$. For a Poisson process, the probability of getting an event in an interval $[t, t + dt]$ is $\lambda dt$. Consequently the probability distribution of waiting times is $p[N_t = 0] \lambda dt = e^{-\lambda t} (\lambda t)^0 / 0!$. Note that this integrates to unity over the range $t \in (0, \infty)$.

We may also solve for the probability density of waiting times by writing and then solving an integral relation. Let $p(t) dt$ be the probability that the waiting time is in $[t, t + dt]$. Then $\int_0^t p(t') dt'$ is the probability that the waiting time is less than $t$. Subtract this from unity to get the probability that the waiting time is at least $t$, $1 - \int_0^t p(t') dt'$. Consider the probability that the waiting time is at least $t$ and an event happens in the following time interval $dt$; these are independent events, so we may just multiply by the probability $\lambda dt$ of an event occurring in a duration $dt$. We’ve recovered an expression for the probability that the waiting time is between $t$ and $t + dt$, giving us the integral relation:

$$\left(1 - \int_0^t p(t') dt'\right) \lambda dt = p(t) dt$$

We can begin to solve this by taking the derivative with respect to $t$. Note that $\int_0^t f(t') dt' = F(t) - F(0)$, where $F'(t) = f(t)$, so $(d/dt) \int_0^t f(t') dt' = f(t)$. We get:

$$p'(t) = \frac{d}{dt} \left(1 - \int_0^t p(t') dt'\right) \lambda = -\lambda p(t)$$

This has the well-known solution

$$p(t) = Ce^{-\lambda t}$$

where $C$ is some constant. The normalization requirement $\int_0^\infty p(t) dt = 1$ gives us

$$p(t) = \lambda e^{-\lambda t}$$

which agrees with what we found earlier.

7.2. Nearest neighbor distances between randomly spaced points Assume that homes in the prairie are distributed uniformly with an average density of $n$ per square mile. What is the probability distribution of the distance to the nearest neighbor from a given home? What is the average distance between nearest neighbors?
This question is similar to 7.1 above; instead of a “waiting time” until the next event after some arbitrary
starting time, we’re interested in the “waiting distance” as we travel radially outward from a given point
until we encounter another house.

Approaching this using the integral relation technique, we may write
\[
\left(1 - \int_0^r p(r')dr'\right)2\pi r dr = p(r) dr
\]
where the density \( n \) fills the role of \( \lambda \) in the one-dimensional case.

Requiring that \( p(0) = 0 \) and \( \int_0^\infty p(r) dr = 1 \) (note the lower limit of integration; the probability of a
negative waiting time is zero), we find
\[
p(r) = 2\pi nr \exp\{-n\pi r^2\}
\]
The average nearest-neighbor distance is given by
\[
\langle r_{\text{nearest neighbor}} \rangle = \int_0^\infty r p(r) dr = \frac{1}{2\sqrt{n}}
\]

8. Transformation of Random Variables

There is a nice explanation of this in *Numerical Recipes in C*, chapter 7.2. The text is available freely at
http://www.library.cornell.edu/nr/bookcpdf/c7-2.pdf.

We employ conservation of probability: \( |p_y(y)dy| = |p_x(x)dx| \). We are given that the probability density
of \( x \) is uniform, i.e. \( p_x(x) = 1 \), and we are given several desired probability densities \( p_y \). The procedure is
to solve for the derivative \( dx/dy \), obtain \( x \) in terms of \( y \) by integrating, and then invert the relation to get
\( y \) in terms of \( x \).

\[
|p_y(y)dy| = |p_x(x)dx| \\
\frac{dx}{dy} = \pm p_y(y) \\
x = \int\frac{dx}{dy}dy = \pm \int p_y(y)dy
\]
The following Mathematica code performs this procedure for \( p_y(y) = -e^{-y} \):
\[
py[y_] := -\text{Exp}[-y] \\
\text{Solve}[x==\text{Integrate}[\text{py}[y],y],y]
\]
We find that \( y_1(x) = -\log(x) \), \( y_2(x) = \text{erf}^{-1}(2x - 1) \), and \( y_3(x) = \tan(\pi x) \).

9. Multiplicative Random Walk Consider the following simple model for the size of a colony of bacteria.
We start with a number \( n_0 \); in each generation the number can be either multiplied by a factor \( u \) with
probability 1/2 or divided by the same number with the same probability. What is the probability distribution
of the number of bacteria after a large number \( N \) of steps? The number \( u \) is near unity.

If \( \eta \) is the random variable giving the population of the colony after many steps, then we may write \( \eta \) as
a product over many random variables \( \eta_i \) each of which may attain the values \( u \) and \( 1/u \), describing how
the size of the colony changes in the \( i \)th step:
\[
\eta = \eta_1\eta_2\eta_3\cdots\eta_N
\]
If we take the logarithm, then the product is converted into a sum:
\[
\log \eta = \log \eta_1 + \log \eta_2 + \log \eta_3 + \cdots + \log \eta_N
\]

The central limit theorem tells us that the sum of many independent random variables will have a normal (Gaussian) distribution. If the logarithm of \( \eta \) follows the normal distribution, then \( \eta \) itself follows the so-called log-normal distribution (see http://en.wikipedia.org/wiki/Log_normal).

One may also write the final size of the colony as \( y = n_0 u^x (1/u)^{N-k} = n_0 u^{2x-N} \), where \( x \) is the random variable giving the number of successes in \( N \) Bernoulli trials; \( k \) will in general follow the binomial distribution with mean \( \mu = Np = N/2 \) and variance \( \sigma^2 = npq = N/4 \), but for large \( N \) this converges to the normal distribution with the same mean and variance.

Use the conservation of probability formula:

\[
|p(y)dy| = |p(x)dx|
\]

to get:

\[
p(y) = \frac{dx}{dy} p(x)
\]

Solve the expression \( y = n_0 u^{2x-N} \) for \( x \):

\[
x = \frac{\log (y/n_0)}{2 \log u} + \frac{N}{2}
\]

Take the derivative:

\[
\frac{dx}{dy} = \frac{1}{2y \log u}
\]

We know that \( p(x) \) is the probability density of the normal distribution with mean \( \mu = N/2 \) and variance \( \sigma^2 = N/4 \):

\[
p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} = \sqrt{\frac{2}{\pi N}} \exp \left\{ \frac{-(N-2x)^2}{2N} \right\} = \sqrt{\frac{2}{\pi N}} \exp \left\{ -\frac{\left( \log \frac{y}{n_0} \right)^2}{2N(\log u)^2} \right\}
\]

So we have \( p(y) = \frac{dx}{dy} p(x) \), which becomes, with everything plugged in:

\[
p(y) = \frac{1}{y(\log u) \sqrt{2\pi N}} \exp \left\{ -\frac{\left( \log \frac{y}{n_0} \right)^2}{2N(\log u)^2} \right\}
\]

Note that this is the probability density function of the log-normal distribution with mean \( \mu = \log(n_0) \) and variance \( \sigma^2 = N(\log u)^2 \).