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1 Problem 22

We can parametrize the components of our unit vector in several ways, such as $\left(v, \sqrt{v^2-1}\right)$ or $(\cos\theta, \sin\theta)$. I'll use the latter. When working with vector components, we can regard bras as (complex-conjugated) row vectors and kets as column vectors. Our function becomes:

$$f(\theta) = \langle\theta|\sigma_1\theta\rangle = \left[\begin{array}{cc} \cos\theta & \sin\theta \end{array}\right] \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right] = \left[\begin{array}{cc} \cos\theta & \sin\theta \end{array}\right] \left[\begin{array}{c} \sin\theta \\ \cos\theta \end{array}\right] = 2\sin(\theta)\cos(\theta)$$

To minimize this function we do the standard drill of setting the first derivative equal to zero and solving for the parameter

$$f'(\theta) = \frac{\partial}{\partial\theta} f(\theta) = -2\sin^2\theta + 2\cos^2\theta$$

$$\sin^2\theta = \cos^2\theta$$

$$\theta_\pm = \pm\frac{\pi}{4}$$

Check the sign of the second derivative to determine whether these extrema are maxima or minima

$$f''(\theta) = \frac{\partial^2}{\partial\theta^2} f(\theta) = -4\sin\theta\cos\theta - 4\cos\theta\sin\theta = -8\sin\theta\cos\theta$$

We find that $f''(\theta_+) < 0$, so θ_+ is a maximum of f, and that $f''(\theta_-) > 0$, so θ_- is a minimum of f. We have:

$$\begin{array}{cc} f(\theta_+) = 1 & f(\theta_-) = -1 \\ |\theta_+\rangle = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) & |\theta_-\rangle = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \end{array}$$

Interpreting the action of σ_1 as swapping the components of the vector it acts upon, we see that $|\theta_\pm\rangle$ are indeed eigenvectors of σ_1 with associated eigenvalues ±1.

2 Problem 23.0

On an inner product space V we may associate with an operator A another operator $A^\dagger$, called the adjoint of A, with the property that $\forall \mathbf{x}, \mathbf{y} \in V$, $\langle A^\dagger\mathbf{x}|\mathbf{y}\rangle = \langle\mathbf{x}|A\mathbf{y}\rangle$.

$$\begin{array}{ll}
\langle\chi|A\psi\rangle = \langle A^\dagger\chi|\psi\rangle & \text{definition of adjoint} \\
\langle\chi|\lambda\psi\rangle = \langle\mu\chi|\psi\rangle & \text{assumption} \\
\lambda\langle\chi|\psi\rangle = \mu^*\langle\chi|\psi\rangle & \text{sesqui-linearity of the inner product} \\
(\lambda - \mu^*)\langle\chi|\psi\rangle = 0 & \\
(\lambda = \mu^*) \vee (\langle\chi|\psi\rangle = 0) & \\
(\lambda \neq \mu^*) \implies (\langle\chi|\psi\rangle = 0) &
\end{array}$$

Note that the vee ($\vee$) indicates logical 'or' (disjunction). Similarly, a wedge ($\wedge$) would indicate logical 'and' (conjunction).

3 Problem 23.1

To show that eigenvalues of a hermitian operator are real, follow the same argument as above, but assume that $A^\dagger = A$ and let $|\chi\rangle = |\psi\rangle$. Furthermore, assume $|\psi\rangle \neq |\mathbf{0}\rangle$ because we don't normally consider the zero vector $|\mathbf{0}\rangle$ as an eigenvector of any operator.

$$\begin{array}{ll}
\langle\psi|A\psi\rangle = \langle A^\dagger\psi|\psi\rangle & \text{definition of adjoint} \\
\langle\psi|A\psi\rangle = \langle A\psi|\psi\rangle & \text{assumption that } A \text{ is self-adjoint} \\
\langle\psi|\lambda\psi\rangle = \langle\lambda\psi|\psi\rangle & \text{assumption that } \psi \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda \\
\lambda\langle\psi|\psi\rangle = \lambda^*\langle\psi|\psi\rangle & \text{sesqui-linearity of the inner product} \\
(\lambda - \lambda^*)\langle\psi|\psi\rangle = 0 & \\
(\lambda = \lambda^*) \vee (\langle\psi|\psi\rangle = 0) & \\
(\lambda \in \mathbb{R}) \vee (\psi\rangle = |\mathbf{0}\rangle) & \text{definiteness of inner product: } \langle\mathbf{x}|\mathbf{x}\rangle = 0 \iff |\mathbf{x}\rangle = |\mathbf{0}\rangle \\
\lambda \in \mathbb{R} &
\end{array}$$

To see that eigenvectors of a hermitian operator with differing eigenvalues are orthogonal, take the conclusion from problem 23, $(\lambda \neq \mu^*) \implies (\langle\chi|\psi\rangle = 0)$. For a hermitian operator, we just showed that the eigenvalues λ and μ must be real, so $\mu^* = \mu$. This gives us the conclusion we desired:

$$(\lambda \neq \mu) \implies (\langle\chi|\psi\rangle = 0)$$

4 Problem 24

Suppose we have a countably infinite set of orthogonal basis vectors $|n\rangle$ for $n \in \mathbb{N}$. Suppose further that we have 'shift' operators A and $A^\dagger$ such that

$$\begin{array}{ll}
A|n\rangle = |n-1\rangle & \text{for } n > 0 \\
A|n\rangle = \mathbf{0} & \text{for } n = 0 \\
A^\dagger|n\rangle = |n+1\rangle &
\end{array}$$

Note that the zero vector is written here just as a bold zero ($\mathbf{0}$) and that the ket labelled by zero ($|0\rangle$) is not (necessarily) equal to the zero vector.

$$\mathbf{0} \neq |0\rangle$$

We're looking for a vector $|z\rangle$ that is an eigenvector of A. Any vector may be written as a linear combination of basis vectors; in particular, our vector $|z\rangle$ may be written that way, where the coefficients depend in some way on z:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n(z)|n\rangle$$

We insist that $|z\rangle$ be an eigenvector of A with eigenvalue z, so

$$A|z\rangle = z|z\rangle$$

On each side, expand $|z\rangle$ as a linear combination of basis vectors:

$$A\sum_{n=0}^{\infty} c_n(z)|n\rangle = z\sum_{n=0}^{\infty} c_n(z)|n\rangle$$

We know how A acts on these basis vectors, so we may perform that operation:

$$\sum_{n=1}^{\infty} c_n(z)|n-1\rangle = \sum_{n=0}^{\infty} zc_n(z)|n\rangle$$

Do a change of variable $n \to n+1$ on the dummy variable in the sum on the left:

$$\sum_{n=0}^{\infty} c_{n+1}(z)|n\rangle = \sum_{n=0}^{\infty} z c_n(z)|n\rangle$$

Equate coefficents of each basis vector on each side:

$$c_{n+1}(z) = z c_n(z)$$

We find that

$$\boxed{c_n(z) = z^n c_0(z)}$$

and our expression for $|z\rangle$ becomes

$$|z\rangle = c_0(z) \sum_{n=0}^{\infty} z^n |n\rangle$$

The squared magnitude of this vector, given the standard norm, is

$$|\mathbf{z}|^2 = \langle z|z\rangle = |c_0(z)|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{|c_0(z)|^2}{1-|z|^2}$$

This is an infinite geometric series, which converges for $|z|^2 < 1$. If we insist that our eigenvector be normalized (of unit length) then we find

$$|c_0(z)|^2 = 1 - |z|^2$$

and our normalized eigenvector has expansion

$$\boxed{|z\rangle = e^{i\phi}\sqrt{1-|z|^2} \sum_{n=0}^{\infty} z^n |n\rangle}$$

up to a choice of phase $\phi \in [0, 2\pi)$.

By contrast, $A^\dagger$ has no eigenvectors. To show this, assume $|k\rangle$ is an eigenvector of $A^\dagger$ with eigenvalue $k \neq 0$, and assume $|n\rangle$ is the lowest n such that $\langle n|k\rangle \neq 0$. Then $k\langle n|k\rangle = \langle n|A^\dagger k\rangle = \langle An|k\rangle = \langle n-1|k\rangle \neq 0$. This violates our assumption that $|n\rangle$ was the lowest n with a nonzero projection on $|k\rangle$, so no such $|k\rangle$ can exist with $A^\dagger|k\rangle = k|k\rangle$, $k \neq 0$. This leaves only the possibility of an eigenvector of $A^\dagger$ with eigenvalue 0, but simple considerations show that only vector satisfying the corresponding eigenvector relation is the zero vector, which is not usually considered an eigenvector.

In quantum mechanics, energy eigenstates of the harmonic oscillator have discrete, evenly spaced energy levels. We define raising and lowering operators $A^\dagger$ and A (respectively), analagous to the shifting operators in this problem but with $A|n\rangle = \sqrt{n}|n-1\rangle$ and $A^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ so that $A^\dagger A|n\rangle = n|n\rangle$. Here $A^\dagger A$ is called the number operator because its eigenvectors are the energy eigenstates and the corresponding eigenvalues give the number of the energy level. Eigenstates of the lowering operator $A^\dagger$ are called coherent states and are important in quantum optics and quantum field theory.