1 August 2004

A family $A_n(\alpha)$ of $n \times n$ matrices, parametrized by $\alpha \in (0, 2\pi)$, is given by

$$A_n(\alpha) = \begin{pmatrix} 0 & e^{i\alpha} & 0 & \cdots & 0 & 0 & 0 \\ e^{-i\alpha} & 0 & e^{i\alpha} & \cdots & 0 & 0 & 0 \\ 0 & e^{-i\alpha} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & \cdots & e^{-i\alpha} & 0 & e^{i\alpha} \\ 0 & 0 & 0 & \cdots & 0 & e^{-i\alpha} & 0 \end{pmatrix}$$

(a) Are the eigenvalues of $A_n(\alpha)$ real? Explain.

(b) Let $P_n(\lambda)$ be the characteristic polynomial of $A_n(\alpha)$. Find a recurrence relation relating $P_n(\lambda)$ to $P_{n-1}(\lambda)$ and $P_{n-2}(\lambda)$.

(c) Show that all A_n with odd n have one eigenvalue in common.

(d) Let $D(\alpha, \beta) = A_n(\alpha)B_n(\beta)$ be the matrix obtained from the product of two martices of this family. Are the eigenvalues of (α, β) real? Explain.