Radiation Transfer and Flux Limited Diffusion

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1 Physics of Radiation Transfer

1.1 Spectral Intensity

Typically when we discuss the radiation field we use the spectral intensity $I(\nu, \mathbf{x}, \Omega)$ which is a function of frequency, position, and direction. This is very similar to the phase space density used in deriving the fluid equations $f(\mathbf{x}, \mathbf{v})$ except that

- light always travels at c, so the velocity dependence is just a direction dependence.
- Furthermore, photons can have different frequencies, so there is an extra dimension to the phase space.
- Instead of storing the phase space density of photons, the spectral intensity is the phase space density of energy flux...

Going between photon number and energy just involves a factor of $h\nu$ and going from energy density to energy flux density just involves a factor of c so we have:

$$I(\nu, \mathbf{x}, \Omega,) = h\nu c f(\nu, \mathbf{x}, \Omega,)$$

This can also be seen by considering the differential energy:

$$\mathrm{d}E = I(\nu, \mathbf{x}, \Omega,)\,\mathrm{d}\nu\,\mathrm{d}\Omega\,\mathrm{d}A\,\mathrm{d}t = h\nu f(\nu, \mathbf{x}, \Omega,)\,\mathrm{d}\nu\,\mathrm{d}\Omega\,\mathrm{d}V$$

where the number of photons traveling normal to the surface $\mathrm{d}A$ that cross the surface $\mathrm{d}A$ in time $\mathrm{d}t$ is just the number of photons in the volume $\mathrm{d}V = \mathrm{d}A\,c\,\mathrm{d}t$ (assuming the photons are headed normal to $\mathrm{d}A$)...

so we also have:

$$\mathrm{d}E = h\nu f(\nu, \mathbf{x}, \Omega,)\,\mathrm{d}\nu\,\mathrm{d}\Omega\,\mathrm{d}A\,c\,\mathrm{d}t$$

which gives:

$$I(\nu, \mathbf{x}, \Omega,) = h\nu c f(\nu, \mathbf{x}, \Omega,)$$

1.2 Deriving the Transport Equation

If we consider the Boltzmann transport equation for photons of a specific frequency f_ν we have

$$\frac{\partial}{\partial t} f_\nu + \mathbf{v} \cdot \nabla f + \mathbf{F} \cdot \nabla_p f = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

Now photons don't experience body forces, always travel at the speed of light, and in general the "collision term" consists of photon emission and absorption... so we have

$$\frac{\partial}{\partial t} f_\nu + c\mathbf{n} \cdot \nabla f_\nu = A_\nu - \chi_\nu f_\nu c$$

where A_ν is the emission rate of photons of frequency ν and the mean free path length is given by $\chi_\nu = \sigma_\nu n$ where σ_ν is the particle scattering cross section and n is the number density of particles.

Now if we multiply through by $h\nu$ we have

$$\frac{\partial}{c\partial t} I_\nu + \mathbf{n} \cdot \nabla I_\nu = \eta_\nu - \sigma_\nu I_\nu$$

where $\eta_\nu = h\nu A_\nu$ is the radiative power.

If we solve the transport equation along a characteristic

$$[\mathbf{x}(s), t(s)] = \left[\mathbf{x0} + \mathbf{n}s, \frac{s}{c}\right]$$

we have

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial x^i}\frac{\partial x^i}{\partial s} + \frac{\partial I_\nu}{\partial t}\frac{\partial t}{\partial s} = \mathbf{n} \cdot \nabla I_\nu + \frac{1}{c}\frac{\partial I_\nu}{\partial t} = \eta_\nu(s) - \chi_\nu(s) I_\nu(s)$$

where $f(s) = f(\mathbf{x}(s), t(s)) = f\left(\mathbf{x_0} + \mathbf{n}s, \frac{s}{c}\right)$ and then we can divide through by $\chi_\nu(s)$ we get

$$\frac{\mathrm{d}I_\nu}{\chi_\nu(s)\mathrm{d}s} = \frac{\eta_\nu(s)}{\chi_\nu(s)} - I_\nu(s) = S_\nu(s) - I_\nu(s)$$

Now if we define $\mathrm{d}\tau_\nu = \chi_\nu(s)\mathrm{d}s$ which gives

$$\tau_\nu(s) = \int_0^s \chi_\nu(s')\mathrm{d}s'$$

and

$$s(\tau_\nu) = \int_0^{\tau_\nu} \frac{1}{\chi_\nu} d\tau'_\nu$$

we can write the transport equation in the simplest form

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = S_\nu(\tau_\nu) - I_\nu(\tau_\nu)$$

although the RHS is now more difficult to evaluate as

$$f(\tau_\nu) = f(s(\tau_\nu)) = f(\mathbf{x}(s(\tau_\nu)), t(s(\tau_\nu)))$$

Also if we include scattering then the source function can depend on the mean radiative flux $\frac{cE}{4\pi}$ and the transport equation becomes an integro-differential equation that must be solved iteratively...

There are also a few important dimensionless numbers to consider (table 1):

Table 1:

$\tau = l\kappa = l/\lambda_p$	$\beta = u/c$
$\tau << 1$	streaming limit
$\tau >> 1$, $\beta\tau << 1$	static diffusion limit
$\tau >> 1$, $\beta\tau >> 1$	dynamic diffusion limit

Table 2: The moments of the specific intensity.

Radiation Energy moments	Corresponding fluid moments
$cE = \int_0^\infty \mathrm{d}\nu \int \mathrm{d}\Omega I(\mathbf{n},\nu)$	$\rho = \int \mathrm{d}\mathbf{v} f(\mathbf{v})$
$\mathbf{F} = \int_0^\infty d\nu \int \mathrm{d}\Omega \mathbf{n} I(\mathbf{n},\nu)$	$\rho\mathbf{v} = \int \mathrm{d}\mathbf{v}\, \mathbf{v} f(\mathbf{v})$
$c\mathbf{P} = \int_0^\infty \mathrm{d}\nu \int \mathrm{d}\Omega\, \mathbf{nn}\, I(\mathbf{n},\nu)$	$\mathbf{P} = \int \mathrm{d}\mathbf{v}\, \mathbf{vv} f(\mathbf{v})$

1.3 Equations of Radiation Hydrodynamics

Some of what follows is taken from
`http://adsabs.harvard.edu/abs/2007ApJ...667..626KKrumholzetal.2007`

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{vv}) = -\nabla P + \mathbf{G}$$
$$\frac{\partial e}{\partial t} + \nabla \cdot [(e + P)\,\mathbf{v}] = cG^0$$
$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = -cG_0$$
$$\frac{1}{c^2}\frac{\partial \mathbf{F}}{\partial t} + \nabla \cdot \mathbf{P} = -\mathbf{G}$$

where the moments of the specific intensity are defined as Table 2:
and the radiation 4-force density is given by

$$cG^0 = \int_0^\infty d\nu \int d\Omega\, [\kappa(\mathbf{n},v)I(\mathbf{n},\nu) - \eta(\mathbf{n},v)]$$
$$c\mathbf{G} = \int_0^\infty d\nu \int d\Omega\, [\kappa(\mathbf{n},v)I(\mathbf{n},\nu) - \eta(\mathbf{n},v)]\,\mathbf{n}$$

If we had a closure relation for the radiation pressure then we could solve this system. For gas particles, collisions tend to produce a Boltzmann Distribution which is isotropic and gives a pressure tensor that is a multiple of the identity tensor. Photons do not "collide" with each other and they all have the same velocity 'c' but in various directions. If the field were isotropic than $P^{ij} = \delta^{ij} 1/3E$ but in general $P^{ij} = f^{ij} E$ where 'f' is the Eddington Tensor.

1.4 Simplifying assumptions

- If the flux spectrum of the radiation is direction-independent then we can write the radiation four-force density in terms of the moments of the radiation field

$$
\begin{aligned}
G^0 = \; & \gamma[\gamma^2\kappa_{0E} + (1-\gamma^2)\kappa_{0F}]E - \gamma\kappa_{0P}a_R T_0^4 \\
& - \gamma(\mathbf{v}\cdot\mathbf{F}/c^2)[\kappa_{0F} - 2\gamma^2(\kappa_{0F} - \kappa_{0E})] \\
& - \gamma^3(\kappa_{0F} - \kappa + 0E)(\mathbf{vv}) : \mathbf{P}/c^2 \\
\mathbf{G} = \; & \gamma\kappa_{0F}(\mathbf{F}/c) - \gamma\kappa_{0P}a_R T_0^4(\mathbf{v}/c) \\
& - \left[\gamma^3(\kappa_{0F} - \kappa_{0E})(\mathbf{v}/c)E + \gamma\kappa_{0F}(\mathbf{v}/c)\cdot\mathbf{P}\right] \\
& + \gamma^3(\kappa_{0F} - \kappa_{0E})\left[2\mathbf{v}\cdot\mathbf{F}/c^3 - (\mathbf{vv}) : \mathbf{P}/c^3\right]\mathbf{v}
\end{aligned}
$$

where

κ_{0P}	comoving-frame Planck function weighted opacity
κ_{0E}	comoving-frame radiation energy weighted opacity
κ_{0F}	comoving-frame radiation flux weighted opacity

- If the radiation has a blackbody spectrum then $\kappa_{0E} = \kappa_{0P}$
- If the radiation is optically thick, then

$$\mathbf{F_0}(\nu_0) \propto -\nabla E_0(\nu_0)/\kappa_0(\nu_0) \propto -[\partial B(\nu_0, T_0)/\partial T_0](\nabla T_0)/\kappa_0(\nu_0)$$

which implies that

$$\kappa_{0F}^{-1} = \kappa_{0R}^{-1} = \frac{\int_0^\infty d\nu_0 \kappa_0(\nu_0)^{-1}[\partial B(\nu_0, T_0)/\partial T_0]}{\int_0^\infty d\nu_0 [\partial B(\nu_0, T_0)/\partial T_0]}$$

- In the optically thin regime, $|\mathbf{F}_0(\nu_0)| \to cE_0(\nu_0)$, so we would have $\kappa_{0F} = \kappa_{0E}$.
  however assuming a blackbody temperature in the optically thin limit may be any more accurate than assuming that $\kappa_{0F} = \kappa_{0R}$

1.5 Flux limited diffusion

The flux limited diffusion approximation drops the radiation momentum equation in favor of

$$\mathbf{F}_0 = -\frac{c\lambda}{\kappa_{0R}}\nabla E_0$$

where λ is the flux-limiter and is given by

$$\lambda = \frac{1}{R}\left(\coth R - \frac{1}{R}\right)$$

and

$$R = \frac{|\nabla E_0|}{\kappa_{0R}E_0}$$

which corresponds to a pressure tensor

$$\mathbf{P}_0 = \frac{E_0}{2}[(1-R_2)\mathbf{I} + (3R_2 - 1)\mathbf{n}_0\mathbf{n}_0]$$

where

$$R_2 = \lambda + \lambda^2 R^2$$

Here are $\lambda(R)$ and $R_2(R)$ plotted

If we Lorentz boost the comoving terms into the lab frame and keep terms necessary to maintain accuracy we get:

$$\begin{aligned}G^0 = &\ \kappa_{0P}\left(E-\frac{4\pi B}{c}\right)+\left(\frac{\lambda}{c}\right)\left(2\frac{\kappa_{0P}}{\kappa_{0R}}-1\right)\mathbf{v}\cdot\nabla E\\ &-\frac{\kappa_{0P}}{c^2}E\left[\frac{3-R_2}{2}v^2+\frac{3R_2-1}{2}(\mathbf{v}\cdot\mathbf{n})^2\right]+\frac{1}{2}\left(\frac{v}{c}\right)^2\kappa_{0P}\left(E-\frac{4\pi B}{c}\right)\\ \mathbf{G} = &-\lambda\nabla E+\kappa_{0P}\frac{\mathbf{v}}{c}\left(E-\frac{4\pi B}{c}\right)-\frac{1}{2}\left(\frac{v}{c}\right)^2\lambda\nabla E+2\lambda\left(\frac{\kappa_{0P}}{\kappa_{0R}}-1\right)\frac{(\mathbf{v}\cdot\nabla E)\mathbf{v}}{c^2}\end{aligned}$$

2 Numerics of Flux Limited Diffusion

If we plug the expressions for the radiation 4-momentum back into the gas equations and keep terms necessary to maintain accuracy we get:

$$\frac{\partial}{\partial t}(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{v}\mathbf{v})=\nabla P-\lambda\nabla E$$

$$\frac{\partial e}{\partial t}+\nabla\cdot[(e+P)\mathbf{v}]=-\kappa_{0P}(4\pi B-cE)+\lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}}-1\right)\mathbf{v}\cdot\nabla E-\frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$$

$$\frac{\partial E}{\partial t}-\nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E=\kappa_{0P}(4\pi B-cE)-\lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}}-1\right)\mathbf{v}\cdot\nabla E-\nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right)+\frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$$

For static diffusion, the terms in blue with $\frac{v^2}{c}$ can be dropped and the system can be split into the usual hydro update (black), radiative source terms (green), and a coupled implicit solve (red) for the radiation energy density and thermal energy density (i.e. temperature).

2.1 Operator splitting

Krumholz et al. perform Implicit Radiative, Explicit Hydro, Explicit Radiative

In AstroBEAR this would look like:

- Initialization
- Prolongate, ρ, $\rho\mathbf{v}$, e, E, and $\dot{E}^I$
- Step 1
- Overlap ρ, $\rho\mathbf{v}$, e, E and do physical BC's
- Do IR which updates e_0 and E_0 using ρ_1, e_1, E_1, and $\dot{E}^I_1$
- Update E_{2mbc} using $\dot{E}^I_{\text{2mbc}}$
- Update e_{2mbc} using E_{2mbc} , $\dot{E}^I_{\text{2mbc}}$, and e_{2mbc}
- Update $\dot{E}^I_0$ using pre IR and post IR E_0
- Ghost e_{2mbc} , $E_{\text{mbc+1}}$, $\dot{E}^I_{\text{mbc+1}}$
- Do first EH,,mbc,,

- Do ER,,mbc,, — Terms with ∇E can be done without ghosting since EH did not change E . The $\nabla \cdot \mathbf{v}E$ term needs time centered face centered velocities which can be stored during the hydro update.
- Store $\dot{E}$ in child arrays to be prolongated
- Step 2
- Overlap ρ , $\rho\mathbf{v}$, e , E and do physical BC's
- Do IR which updates e_0 and E_0 using ρ_1 , e_1 , E_1 , and $\dot{E}_1^I$
- Update $\dot{E}_0^I$ using pre IR and post IR E_0
- Update E_1 using $\dot{E}_1^I$
- Ghost e_{mbc} , E_1 , $\dot{E}_1^I$
- Do second EH,,0,,
- Do ER,,mbc,, — Terms with ∇E can be done without ghosting since EH did not
- Do ER,,0,, — Terms with grad E can be done without ghosting since EH did not change E. The $\nabla \cdot \mathbf{v}E$ term needs time centered face centered velocities which can be stored during the hydro update.

2.2 Explicit Update 1

The extra terms in the explicit update due to radiation energy are as follows:

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = -\lambda \nabla E$$

$$\frac{\partial e}{\partial t} = \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E$$

$$\frac{\partial E}{\partial t} = -\lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right)$$

These can be discretized as follows:

$$p_i^{n+1} = p_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\lambda_i\left(E_{i+1}^n - E_{i-1}^n\right)$$

$$e_i^{n+1} = e_i^n + \frac{1}{2}\frac{\Delta t}{\Delta x}\lambda_i\left(2\frac{\kappa_{i,0P}^n}{\kappa_{i,0R}^n} - 1\right)\left(v_i^n\left(E_{i+1}^n - E_{i-1}^n\right)\right)$$

$$E_i^{n+1} = E_i^n - \frac{\Delta t}{\Delta x}\left(\frac{\lambda_i}{2}\left(2\frac{\kappa_{i,0P}^n}{\kappa_{i,0R}^n} - 1\right)\left(v_i^n\left(E_{i+1}^n - E_{i-1}^n\right)\right) + \left(F_{i+1/2} - F_{i-1/2}\right)\right)$$

where

$$F_{i+1/2} = \frac{3 - R_{2,i+1/2}}{8}\left(v_i^n + v_{i+1}^n\right)\left(E_i^n + E_{i+1}^n\right)$$

where

$$R_{2,i+1/2} = \lambda_{i+1/2} + \lambda_{i+1/2}^2 R_{i+1/2}^2$$

and

$$R_{i+1/2} = \frac{\left|E_{i+1}^n - E_i^n\right|}{2\kappa_{0R,i+1/2}\left(E_i^n + E_{i+1}^n\right)}$$

and

$$\lambda_{i+1/2} = \frac{1}{R_{i+1/2}} \left(\coth R_{i+1/2} - \frac{1}{R_{i+1/2}} \right)$$

and

$$\kappa_{0R,i+1/2} = \frac{\kappa^n_{0R,i} + \kappa^n_{0R,i+1}}{2} \text{ and } \lambda_{i+1/2} = \frac{1}{R_{i+1/2}} \left(\coth R_{i+1/2} - \frac{1}{R_{i+1/2}} \right)$$

and

$$\lambda_i = \frac{1}{R_i} \left(\coth R_i - \frac{1}{R_i} \right)$$

and

$$R_i = \frac{\left| E^n_{i+1} - E^n_{i-1} \right|}{2\kappa_{0R,i} E^n_i}$$

2.3 Implicit Update 1

For now we will assume that κ_{0P} and κ_{0R} are constant over the implicit update and we will treat the energy as the total internal energy ignoring kinetic and magnetic contributions. In this case we can solve the radiation energy equations:

$$\frac{\partial E}{\partial t} = \nabla \cdot \frac{c\lambda}{\kappa_{0R}} \nabla E + \kappa_{0P}(4\pi B(T) - cE) = \nabla \cdot \mathbf{F} + \kappa_{0P}(4\pi B(T) - cE)$$
$$\frac{\partial e}{\partial t} = -\kappa_{0P}(4\pi B(T) - cE)$$

where $\mathbf{F} = \frac{c\lambda}{\kappa_{0R}} \nabla E$

Expanding about e,,0,,

Of course even if the opacity is independent of energy and radiation energy, the above combined system of equations is still non-linear due to the dependence on Temperature of the Planck Function $B(T)$

However we can expand the Plank Function about e_0

$$B(T) = B(T_0 + dT) = B(T_0) + \left. \frac{\partial B}{\partial T} \right|_{T_0} \frac{\partial T}{\partial e} de = \frac{c}{4\pi} a_R \left(T_0^4 + 4T_0^3 \Gamma de \right) = B_0 \left(1 + 4\Gamma \frac{e - e_0}{T_0} \right)$$

where

$\Gamma = \frac{\partial T}{\partial e} = \frac{(\gamma - 1)}{nk_B}$

Then the system of equations becomes

$\frac{\partial E}{\partial t} = \nabla \cdot \frac{c\lambda}{\kappa_{0R}} \nabla E + \kappa_{0P} \left[4\pi B_0 \left(1 + 4\Gamma \frac{e - e_0}{T_0} \right) - cE \right]$ $\frac{\partial e}{\partial t} = -\kappa_{0P} \left[4\pi B_0 \left(1 + 4\Gamma \frac{e - e_0}{T_0} \right) - cE \right]$

which will be accurate as long as $\left| 4\Gamma \frac{e - e_0}{T_0} \right| < \xi << 1$ or $\Delta e = |e - e_0| < \xi \frac{T_0}{4\Gamma}$

We can calculate the time scale for this to be true using the evolution equation for the energy density

$\Delta e = \Delta t \kappa_{0P} \left| 4\pi B_0 - cE \right| < \xi \frac{T_0}{4\Gamma}$

which gives $\Delta t < \xi \frac{T_0}{4\Gamma \kappa_{0P} |4\pi B_0 - cE|}$

Implicit Discretization 1

We can now discretize the equations

$E^{n+1}_i - E^n_i = \left[\alpha_{i+1/2} \left(E^*_{i+1} - E^*_i \right) - \alpha_{i-1/2} \left(E^*_i - E^*_{i-1} \right) \right] - \epsilon E^*_i + \phi e^*_i + \theta_i - \phi_i e^n_i)$ $e^{n+1}_i - e^n_i = \epsilon_i E^*_i - \phi_i e^*_i - (\theta_i - \phi_i e^n_i)$

where the diffusion coefficient is given by

$\alpha_{i+1/2} = \frac{\Delta t}{\Delta x^2} \frac{c\lambda_{i+1/2}}{\kappa_{0R,i+1/2}}$ where $\kappa_{0R,i+1/2} = \frac{\kappa^n_{0R,i} + \kappa^n_{0R,i+1}}{2}$ and $\lambda_{i+1/2} = \frac{1}{R_{i+1/2}} \left(\coth R_{i+1/2} - \frac{1}{R_{i+1/2}} \right)$ and where $R_{i+1/2} = \frac{\left| E^n_{i+1} - E^n_i \right|}{2\kappa_{0R,i+1/2} \left(E^n_i + E^n_{i+1} \right)}$

and
$\epsilon_i = c\Delta t \kappa_{0P,i}$
represents the number of absorption/emissions during the time step
and
$\phi_i = \epsilon_i \frac{4\pi}{c} B\left(T_i^n\right)\left(\frac{4\Gamma}{T_i^n}\right)$
$\theta_i = \epsilon_i \frac{4\pi}{c} B\left(T_i^n\right)$
and we can think of the radiative flux as
$\frac{\Delta t}{\Delta x}\mathbf{F}_{i+1/2} = \alpha_{i+1/2}\left(E_{i+1}^* - E_i^*\right)$

Time Discretization

Now all the terms on the right hand side that are linear in E or e have been written as E^* or e^* because there are different ways to approximate $E^{*(}e^{*)}$. For Backward Euler we have $E_i^* = E_i^{n+1}$ and for Crank Nicholson we have $E_i^* = \frac{1}{2}\left(E_i^{n+1} + E_i^n\right)$ or we can parameterize the solution $E_i^* = \psi E_i^{n+1} + \bar{\psi} E_i^n$ where $\bar{\psi} = 1 - \psi$

Backward Euler has $\psi = 1$ and Crank Nicholson has $\psi = 1/2$
Forward Euler has $\psi = 0$
In any event in 1D we have the following matrix coefficients

$$\begin{gathered}\left[1 + \psi\left(\alpha_{i+1/2} + \alpha_{i-1/2} + \epsilon_i\right)\right] E_i^{n+1} - \left(\psi\alpha_{i+1/2}\right) E_{i+1}^{n+1} - \left(\psi\alpha_{i-1/2}\right) E_{i-1}^{n+1} - \left(\psi\phi_i\right) e_i^{n+1} \\ = \left[1 - \bar{\psi}\left(\alpha_{i+1/2} + \alpha_{i-1/2} + \epsilon_i\right)\right] E_i^n + \left(\bar{\psi}\alpha_{i+1/2}\right) E_{i+1}^n + \left(\bar{\psi}\alpha_{i-1/2}\right) E_{i-1}^n - \psi\phi_i e_i^n + \theta_i \\ \left(1 + \psi\phi_i\right) e_i^{n+1} - \left(\psi\epsilon_i\right) E_i^{n+1} = \left(1 + \psi\phi_i\right) e_i^n + \left(\bar{\psi}\epsilon_i\right) E_i^n - \theta_i\end{gathered}$$

Now since the second equation has no spatial dependence, we can solve it for

$$e_i^{n+1} = e_i^n + \frac{1}{1 + \psi\phi_i}\left\{\left(\psi\epsilon_i\right) E_i^{n+1} + \left(\bar{\psi}\epsilon_i\right) E_i^n - \theta_i\right\}$$

and plug the result into the first equation to get a matrix equation involving only one variable.

$$\begin{aligned}\left[1 + \psi\left(\alpha_{i+1/2} + \alpha_{i-1/2} + \frac{\epsilon_i}{1 + \psi\phi_i}\right)\right] E_i^{n+1} - \left(\psi\alpha_{i+1/2}\right) E_{i+1}^{n+1} - \left(\psi\alpha_{i-1/2}\right) E_{i-1}^{n+1} \\ = \left[1 - \bar{\psi}\left(\alpha_{i+1/2} + \alpha_{i-1/2} + \frac{\epsilon_i}{1 + \psi\phi_i}\right)\right] E_i^n + \left(\bar{\psi}\alpha_{i+1/2}\right) E_{i+1}^n + \left(\bar{\psi}\alpha_{i-1/2}\right) E_{i-1}^n + \frac{\theta_i}{1 + \psi\phi_i}\end{aligned}$$

This equation decouples and can be solved on it's own, and then the solution plugged back into the second equation to solve for the new energy.

2D etc...

For 2D or 3D we have more connections to add to the matrix elements but it is very straight forward... There will be additional alpha terms for each dimension, but everything else stays the same.

Initial solution vector For the initial solution vector, we can just use Edot from the parent update (or last time step if we are on the coarse grid) to guess E, and then we can solve for the new e given our guess at the new E using the same time stepping (Backward Euler, Crank Nicholson, etc...).
$E_i^{n+1} = E_i^n + \dot{E}_i^I \Delta t$ $e_i^{n+1} = e_i^n + \frac{1}{1+\psi\phi_i}\left\{\left(\psi\epsilon_i\right) E_i^{n+1} + \left(\bar{\psi}\epsilon_i\right) E_i^n - \theta_i\right\}$

2.3.1 Coarse-Fine Boundaries

Since we are doing our implicit solves first, we can use time interpolated solutions for the implicit solve for non-refined ghost zones. To do this we just need Edot. The opacities etc... in the ghost zones can be obtained from the hydro terms.

And the radiative implicit heating in coarse ghost cells can be done along with the initial solution vector so they are available for the hydro update.

2.3.2 Physical Boundary Conditions

Open (Free streaming) boundaries We would like the radiation to leave at the free streaming limit. So $\frac{c\lambda}{\kappa_{0R}}\nabla E = \mathbf{F} = cE\mathbf{n} = \frac{c\lambda_g}{\kappa_g}\frac{(E-E_g)}{\Delta x}$

Clearly if we set $E_g = 0$ and $\lambda_g = \kappa_g \Delta x$ we should get the correct flux.

This corresponds to an $\alpha = c\frac{\Delta t}{\Delta x}$

So we would just modify α and zero out the matrix coefficient to the ghost zone

Constant Slope Boundary Here we want the flux to be constant so energy does not pile up near the boundary. If we cancel all derivative terms on both sides of the cell, this will effectively match the incoming flux with the outgoing flux. This can also be done by setting $\alpha_g = \alpha_i = 0$

Periodic Boundary This is the same as internal zones - it just maps the neighbor cell to be across the domain. Hypre has built in functionality for this under for the Struct Interface

User defined radiation field/Coarse Fine boundary

This will be the boundary at internal coarse-fine boundaries, but could also be used at the physical boundary if the radiation energy were specified.

Reflecting/ZeroSlope Boundary Reflecting boundary should be fairly straightforward. This an be achieved by setting $\alpha_g = 0$ which zeros out any flux - and has the same effect as setting $E_g^* = E_i^*$ or $E_g^{n+1} = E_i^{n+1}$ and $E_g^n = E_i^n$

Constant radiative flux To have a constant radiative flux we must zero out terms involving the gradient and just add $F_0\frac{\Delta t}{\Delta x}$ in the source vector

Summary Table 3

Table 3: Summary

Numerical value	Boundary	E_{i+1}^{n+1}	E_i^{n+1}	E_{i-1}^{n+1}	S
0	RAD_FREE_STREAMING	0	$\psi c \frac{\Delta t}{\Delta x}$	$-\psi\alpha_{i-1/2}$	$-\bar{\psi} c \frac{\Delta t}{\Delta x} E_i^n + \bar{\psi}\alpha_{i-1/2} \left(E\right.$
1	RAD_EXTRAPOLATE_FLUX	0	0	0	0
2	INTERNALPERIODIC	$-\psi\alpha_{i+1/2}$	$\psi\alpha_{i+1/2}$	$-\psi\alpha_{i-1/2}$	$\bar{\psi}\alpha_{i+1/2}\left(E_{i+1}^n - E_i^n\right) + \bar{\psi}\alpha_{i-}$
3	RAD_REFLECTING	0	0	$-\psi\alpha_{i-1/2}$	$\bar{\psi}\alpha_{i-1/2}\left(E_{i-1}^n - \right.$
4	RAD_USERDEFINED_BOUNDARY/AMR_BOUNDARY	0	$\psi\alpha_{i+1/2}$	$-\psi\alpha_{i-1/2}$	$\bar{\psi}\alpha_{i+1/2}\left(E_{i+1}^n - E_i^n\right) - \psi\alpha_{i+1/2}E_{i+1}^{n+1}$
5	RAD_USERDEFINED_FLUX	0	0	$-\psi\alpha_{i-1/2}$	$F_0 \frac{\Delta t}{\Delta x} + \bar{\psi}\alpha_{i-1/2}\left(E_{i-}^n\right.$

2.4 Tightly Coupled Simplifications

If we plug the expressions for the radiation 4-momentum back into the gas equations and keep terms necessary to maintain accuracy we get:

$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla P - \lambda \nabla E$

$\frac{\partial e}{\partial t} + \nabla \cdot [(e+P)\mathbf{v}] = -\kappa_{0P}(4\pi B - cE) + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$

$\frac{\partial E}{\partial t} - \nabla \cdot \frac{c\lambda}{\kappa_{0R}}\nabla E = \kappa_{0P}(4\pi B - cE) - \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right) + \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$

Now if

$E = a_R T^4 = a_R\left(\frac{e}{\rho c_v}\right)^4$

and

$e = \rho c_v T = \rho c_v \left(\frac{E}{a_R}\right)^{1/4}$

and we just consider the implicit terms, we can combine the gas and radiation diffusion equations to arrive at:

$\frac{\partial(e+E)}{\partial t} = \nabla \cdot \frac{c\lambda}{\kappa_{0R}}\nabla E$

which simplifies to

$\left(1 + \frac{\rho c_v}{4E}\left(\frac{E}{a_R}\right)^{1/4}\right)\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E$

or

$\left(1+\frac{e}{4E}\right)\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E$

The second term in parenthesis represents the extra 'inertia' the radiation field has due to its coupling with the gas. It is non-linear and this limits the time step that can be taken.

$\Delta t \approx \frac{E}{\frac{\partial E}{\partial t}} = \frac{E+\frac{e}{4}}{\nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E}$

2.4.1 Changes to the discretization

For the coupled system of equations we had the following:

$$\left[1+\psi\left(\alpha_{i+1/2}+\alpha_{i-1/2}+\frac{\epsilon_i}{1+\psi\phi_i}\right)\right]E_i^{n+1} - \left(\psi\alpha_{i+1/2}\right)E_{i+1}^{n+1} - \left(\psi\alpha_{i-1/2}\right)E_{i-1}^{n+1}$$
$$= \left[1-\bar{\psi}\left(\alpha_{i+1/2}+\alpha_{i-1/2}+\frac{\epsilon_i}{1+\psi\phi_i}\right)\right]E_i^n + \left(\bar{\psi}\alpha_{i+1/2}\right)E_{i+1}^n + \left(\bar{\psi}\alpha_{i-1/2}\right)E_{i-1}^n + \frac{\theta_i}{1+\psi\phi_i}$$

If the gas and radiation are in thermal equilibrium, then we have $\theta_i = \epsilon_i E_i$ and we also have that in the limit that $\kappa_P \to \infty$, we have $\epsilon \to \infty$ and $\phi \to \infty$

This simplifies the above equation to

$$\left[\left(1+\frac{\epsilon_i}{\phi_i}\right)+\psi\left(\alpha_{i+1/2}+\alpha_{i-1/2}\right)\right]E_i^{n+1} - \left(\psi\alpha_{i+1/2}\right)E_{i+1}^{n+1} - \left(\psi\alpha_{i-1/2}\right)E_{i-1}^{n+1}$$
$$= \left[\left(1+\frac{\epsilon_i}{\phi_i}\right)-\bar{\psi}\left(\alpha_{i+1/2}+\alpha_{i-1/2}\right)\right]E_i^n + \left(\bar{\psi}\alpha_{i+1/2}\right)E_{i+1}^n + \left(\bar{\psi}\alpha_{i-1/2}\right)E_{i-1}^n$$

And $\frac{\epsilon_i}{\phi_i} = \frac{\frac{T_i}{4\Gamma}}{E_i} = \frac{\frac{T_i}{4\frac{\partial T}{\partial e}}}{E_i}$ which if we use our equation of state where $e \propto T$ gives $\frac{\epsilon_i}{\phi_i} = \frac{e_i}{4E_i}$

Now if we go back and calculate $\frac{\partial e}{\partial E} = \frac{\partial e}{\partial T}\frac{\partial T}{\partial E} = \frac{\frac{1}{\Gamma}}{\frac{4E}{T}}$ we arrive at $\frac{\epsilon_i}{\phi_i}$ instead of $\frac{e_i}{4E_i}$ which is consistent with our derivation above.

Also our time equation should be

$\Delta t \approx \frac{E}{\frac{\partial E}{\partial t}} = \frac{E+\frac{T_i}{4\Gamma}}{\nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E}$

So in principle combining the gas and energy equations in the limit that of high planck opacity, does not change the matrix or rhs vector, however it does limit the ability for there to be strong source terms on the right in regions where the gas and radiation have gotten out of equilibrium. It is not clear how this effects the ability of the elliptic solver to converge to given tolerances.

2.5 Modifications to time steps

More importantly is the recognition of the time scales over which the internal energy can change.

Previously we looked at the decoupled equation for the gas energy density
$\frac{\partial e}{\partial t} = -\kappa_{0P}(4\pi B - cE)$
$\Delta e = \Delta t \kappa_{0P} \left|4\pi B_0 - cE\right| < \xi \frac{T_0}{4\Gamma}$
which gives $\Delta t < \xi \frac{T_0}{4\Gamma\kappa_{0P}|4\pi B_0 - cE|}$
however, if the gas is in equilibrium with the radiation, this does not limit the time step at all - even though diffusion may quickly move the gas out of equilibrium with the radiation.

We can account for this by expanding our equation
$\Delta t < \xi \frac{T_0}{4\Gamma\kappa_{0P}|4\pi B_0 - c(E_0 + \partial_t E \Delta t)|}$ which gives us a quadratic for Δt
$4\Gamma\kappa_{0P} c \left|\partial_t E\right| \Delta t^2 + \left(4\Gamma\kappa_{0P} \left|4\pi B_0 - cE_0\right|\right) \Delta t < \xi T_0$
where we have conservatively assumed that the diffusion always causes the gas to be more out of equilibrium.

Now if we recognize that there are three time scales at play here:

$$\begin{aligned}
\text{Diffusion Time} \quad \tau_D &= \frac{T}{4\Gamma\left|\nabla \cdot \frac{c\lambda}{\kappa_{0R}} \nabla E\right|} \\
\text{Coupling Time} \quad \tau_C &= \frac{T}{4\Gamma\kappa_{0P}\left|4\pi B_0 - cE\right|} \\
\text{Absorption Time} \quad \tau_A &= \frac{1}{c\kappa_{0P}}
\end{aligned}$$

then this quadratic simplifies to
$\frac{\Delta t^2}{\tau_A \tau_D} + \frac{\Delta t}{\tau_C} < \xi$
$\Delta t = \sqrt{\left(\frac{\tau_A \tau_D}{2\tau_C}\right)^2 + \xi \tau_D \tau_A} - \frac{\tau_A \tau_D}{2\tau_C}$
$\Delta t = \frac{\tau_A \tau_D}{2\tau_C}\left(\sqrt{1 + \frac{4\xi\tau_C^2}{\tau_D \tau_A}} - 1\right)$
So we can choose * the diffusion time if we assume the gas and radiation are strongly coupled - optically thick, * the coupling time if we assume that the gas is optically thin (which should imply the radiation is fairly diffused) * Or we can solve the quadratic

2.6 Alternative Splitting Method

While the previous method for splitting the equation technically works, it is likely to lead to large differences in the radiation field and the energy fields at coarse fine boundaries since there is no radiative flux to coarsen from the explicit updates to keep the various AMR levels consistent. There is one term that could potentially be coarsened and applied like any flux, but it is not the dominant term. As a result it might be better to treat the entire radiative energy equation implicitly (though keeping the velocity constant)

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\nabla P - \lambda \nabla E$$

$$\frac{\partial e}{\partial t} + \nabla \cdot [(e+P)\mathbf{v}] = -\kappa_{0P}(4\pi B - cE) + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$$

$$\frac{\partial E}{\partial t} - \nabla \cdot \frac{c\lambda}{\kappa_{0R}}\nabla E = \kappa_{0P}(4\pi B - cE) - \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right) + \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$$

It is possible that with included the terms in magenta in a semi-implicit method, the dynamic diffusion regime may be stable... In any event, it costs very little to add all of the terms in magenta to the implicit solve (using the old velocity). Then the momentum update can be done explicitly - though using a time centered radiation field.

Operator Splitting 2

2.6.1 Operator splitting 2

In AstroBEAR this would look like: * Initialization * Prolongate, ρ , $\rho\mathbf{v}$, e , E , and $\dot{E}$ * Step 1 * Overlap ρ , $\rho\mathbf{v}$, e , E and do physical BC's * Do IR which updates e_0 and E_0 using ρ_1 , e_1 , E_1 , and $\dot{E}_1$ * Update E_{2mbc} using $\dot{E}_{\text{2mbc}}$ * Update e_{2mbc} using E_{2mbc} , $\dot{E}_{\text{2mbc}}$, and e_{2mbc} * Update $\dot{E}_0$ using pre IR and post IR E_0 * Ghost e_{2mbc} , $E_{\text{mbc+1}}$, $\dot{E}_{\text{mbc+1}}$ * Do first EH,,mbc,, * Do ER,,mbc,, — Terms with ∇E can be done without ghosting since EH did not change E . * Store $\dot{E}$ in child arrays to be prolongated * Step 2 * Overlap ρ , $\rho\mathbf{v}$, e , E and do physical BC's * Do IR which updates e_0 and E_0 using ρ_1 , e_1 , E_1 , and $\dot{E}_1$ * Update $\dot{E}_0$ using pre IR and post IR E_0 * Update E_1 using $\dot{E}_1$ * Ghost e_{mbc} , E_1 , $\dot{E}_1$ * Do second EH,,0,, * Do ER,,0,, — Terms with ∇E can be done without ghosting since EH did not change E .

2.7 Explicit Update 2

The extra terms in the explicit update due to radiation energy are as follows:

$\frac{\partial}{\partial t}(\rho\mathbf{v}) = -\lambda\nabla E$

These can be discretized as follows:

$p_i^{n+1} = p_i^n - \frac{1}{4}\frac{\Delta t}{\Delta x}\lambda_i\left(\left(E_{i+1}^n + E_{i+1}^{n+1}\right) - \left(E_{i-1}^n + E_{i-1}^{n+1}\right)\right)$

$\lambda_i = \frac{1}{R_i}\left(\coth R_i - \frac{1}{R_i}\right)$

and $R_i = \frac{\left|E_{i+1}^n - E_{i-1}^n\right|}{2\kappa_{0R,i}E_i^n}$

2.8 Implicit Update 2

For now we will assume that κ_{0P} and κ_{0R} are constant over the implicit update and we will treat the energy as the total internal energy ignoring kinetic and magnetic contributions. In this case we can solve the radiation energy equations:

$\frac{\partial e}{\partial t} = -\kappa_{0P}(4\pi B - cE) + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$ $\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E + \kappa_{0P}(4\pi B(T) - cE) - \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E + \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right)$

which we can also write as

$\frac{\partial e}{\partial t} = f(e) + g(E,\nabla E)$ $\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right) - f(e) - g(e,E,\nabla E)$ where $f(e) = -4\pi\kappa_{0P}B(T(e))$ and $g(e,E,\nabla E) = \kappa_{0P}(cE) + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$

Now we can linearize f about e,,0,, $f(e) = f(e_0) + \frac{\partial f}{\partial e}(e - e_0)$

so that the first equation can be written as

$\frac{\partial e}{\partial t} = f(e_0) + g(E, \nabla E) + \frac{\partial f}{\partial e}(e - e_0)$
and then discretized as
$e_i^{n+1} - e_i^n = \Delta t\left(f(e_i^n) + g(E^*, \nabla E^*)\right) + \psi\phi e_i^n - \psi\phi e_i^{n+1}$
where
$\phi = -\Delta t\frac{\partial f}{\partial e} = 4\pi\kappa_{0P}\Delta t\frac{\partial B(T(e))}{\partial e}$
which can be solved for $e_i^{n+1} = e_i^n + \frac{\Delta t}{1+\psi\phi}\left(f(e_i^n) + g(E^*, \nabla E^*)\right)$
or
$e_i^{n+1} = e_i^n + \frac{1}{1+\psi\phi_i}\left(-\theta_i + \epsilon_i E_i^* + \omega_{x,i}v_{x,i}^n\left(E_{i+1}^* - E_{i-1}^*\right) - \xi_i E^*\right)$
Then if we take the semi-discretized equation for E
$\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right) - f(e_i^n) - g(E, \nabla E) - \frac{1}{\Delta t}\left(\psi\phi_i e_i^n - \psi\phi_i e_i^{n+1}\right)$
and then plugin the solution for $e^n + 1^, i, ,$ we get
$\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right) - f(e_i^n) - g(E, \nabla E) - \frac{1}{\Delta t}\left(\psi\phi_i e_i^n - \psi\phi_i e_i^n - \frac{\psi\phi_i}{1+\psi\phi_i}\left(f(e_i^n) + g(E, \nabla E)\right)\right)$
which simplifies to
$\frac{\partial E}{\partial t} = \nabla\cdot\frac{c\lambda}{\kappa_{0R}}\nabla E - \nabla\cdot\left(\frac{3-R_2}{2}\mathbf{v}E\right) - \frac{1}{1+\psi\phi_i}\left(f(e_i^n) + g(E, \nabla E)\right)$
Now we have 1 equation with 1 variable that we can solve implicitly using hypre, and then we can use E^{n+1} and E^n to construct E^* which we can plug into the equation for e^{n+1}

Expanding f about e,,0,,

Expanding
$B(T(e)) = B(T_0 + dT(e)) = B(T_0) + \frac{\partial B}{\partial T}\Big|_{T_0}\frac{\partial T}{\partial e}de = \frac{c}{4\pi}a_R\left(T_0^4 + 4T_0^3\Gamma de\right) = B_0\left(1 + 4\Gamma\frac{e-e_0}{T_0}\right)$
where
$\Gamma = \frac{\partial T}{\partial e} = \frac{(\gamma-1)}{nk_B}$
and we can identify $\phi = -\Delta t\frac{\partial f}{\partial e} = 4\pi\kappa_{0P}\Delta t\frac{\partial B}{\partial e} = 16\pi\kappa_{0P}\Delta t B_0\frac{\Gamma}{T_0}$
Then the equation for e becomes
$\frac{\partial e}{\partial t} = -\kappa_{0P}\left[4\pi B_0\left(1 + 4\Gamma\frac{e-e_0}{T_0}\right) - cE\right] + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E$
which will be accurate as long as $4\Gamma\frac{|e-e_0|}{T_0} < \xi << 1$ or $|\Delta e| = |e - e_0| < \xi\frac{T_0}{4\Gamma}$
We can calculate the time scale for this to be true using the evolution equation for the energy density
$|e - e_0| = |\Delta e| = \Delta t\left|\frac{\partial e}{\partial t}\right| < \xi\frac{T_0}{4\Gamma}$
which gives $\Delta t < \xi\frac{T_0}{4\Gamma\left|\frac{\partial e}{\partial t}\right|} = \xi\frac{T_0}{4\Gamma\left|-\kappa_{0P}\left[4\pi B_0\left(1+4\Gamma\frac{e-e_0}{T_0}\right) - cE\right] + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E\right|}$
or in discretized form
$\Delta t < \xi\frac{\theta_i}{\phi_i\frac{1}{\Delta t}\left|\left(-\theta_i + \epsilon_i E_i^* + \omega_{x,i}v_{x,i}^n\left(E_{i+1}^* - E_{i-1}^*\right) - \xi_i E^*\right)\right|}$

Implicit Discretization 2 Now we can discretize $g(E*, \nabla E*) = \kappa_{0P}cE^* + \lambda\left(2\frac{\kappa_{0P}}{\kappa_{0R}} - 1\right)\mathbf{v}\cdot\nabla E^* - \frac{3-R_2}{2}\kappa_{0P}\frac{v^2}{c}E^*$
as
$g = \epsilon_i\left(\psi E_i^{n+1} + \bar{\psi}E_i^n\right) + \omega_{x,i}\left(\psi E_{i+1}^{n+1} - \psi E_{i-1}^{n+1} + \bar{\psi}E_{i+1}^n - \bar{\psi}E_{i-1}^n\right) - \xi_i\left(\psi E_i^{n+1} + \bar{\psi}E_i^n\right)$
which along with the other terms gives

$$\begin{aligned} E_i^{n+1} - E_i^n = &\left[\alpha_{i+1/2}\left(\psi E_{i+1}^{n+1} + \bar{\psi}E_{i+1}^n - \psi E_i^{n+1} - \bar{\psi}E_i^n\right) - \alpha_{i-1/2}\left(\psi E_i^{n+1} + \bar{\psi}E_i^n - \psi E_{i-1}^{n+1} - \bar{\psi}E_{i-1}^n\right)\right] \\ &- \left[\zeta_{i+1/2}v_{x,i+1/2}^n\left(\psi E_{i+1}^{n+1} + \bar{\psi}E_{i+1}^n + \psi E_i^{n+1} + \bar{\psi}E_i^n\right) - \zeta_{i-1/2}v_{x,i-1/2}^n\left(\psi E_i^{n+1} + \bar{\psi}E_i^n + \psi E_{i-1}^{n+1} + \bar{\psi}E_{i-1}^n\right)\right] \\ &- \frac{1}{1+\psi\phi_i}\left[-\theta_i + \epsilon_i\left(\psi E_i^{n+1} + \bar{\psi}E_i^n\right) + \omega_{x,i}v_{x,i}^n\left(\psi E_{i+1}^{n+1} + \bar{\psi}E_{i+1}^n - \psi E_{i-1}^{n+1} - \bar{\psi}E_{i-1}^n\right) - \xi_i\left(\psi E_i^{n+1} + \bar{\psi}E_i^n\right)\right] \end{aligned}$$

where the diffusion coefficient is given by
$\alpha_{i+1/2} = \frac{\Delta t}{\Delta x^2}\frac{c\lambda_{i+1/2}}{\kappa_{0R,i+1/2}}$
and where
$\zeta_{i+1/2} = \frac{\Delta t}{\Delta x}\frac{3-R_{2,i+1/2}}{4}$
and
$\epsilon_i = c\Delta t\kappa_{0P,i}$

and
$\phi_i = \Delta t 4\pi\kappa_{0P,i} B\left(T_i^n\right)\left(\frac{4\Gamma}{T_i^n}\right)$
and
$\theta_i = -\Delta t f(e_i^n) = \Delta t 4\pi\kappa_{0P,i} B\left(T_i^n\right)$
and
$\omega_{x,i} = \frac{\lambda_{x,i}\Delta t}{\Delta x}\left(\frac{\kappa_{0P,i}}{\kappa_{0R,i}} - \frac{1}{2}\right)$
and
$\xi_i = \Delta t \frac{3-R_{2,i}}{2}\kappa_{0P,i}\frac{v_i^2}{c}$ and
where $\kappa_{0R,i+1/2} = \frac{\kappa_{0R,i}+\kappa_{0R,i+1}}{2}$
and
$R_{i+1/2} = \frac{\left|E_{i+1}^n - E_i^n\right|}{2\kappa_{0R,i+1/2}\left(E_i^n + E_{i+1}^n\right)}$
and
$\lambda_{i+1/2} = \frac{1}{R_{i+1/2}}\left(\coth R_{i+1/2} - \frac{1}{R_{i+1/2}}\right)$
and
$R_{2,i+1/2} = \lambda_{i+1/2} + \lambda_{i+1/2}^2 R_{i+1/2}^2$
and
$R_i = \frac{\left|E_{i+1}^n - E_{i-1}^n\right|}{2\kappa_{0R,i}E_i^n}$
and
$\lambda_i = \frac{1}{R_i}\left(\coth R_i - \frac{1}{R_i}\right)$
and
$R_{2,i} = \lambda_i + \lambda_i^2 R_i^2$
Which we can arrange into the following form

$$\left(1+\psi\left(\alpha_{i+1/2}+\alpha_{i-1/2}+\zeta_{i+1/2}v_{x,i+1/2}^n-\zeta_{i-1/2}v_{x,i-1/2}^n+\frac{\epsilon_i-\xi_i}{1+\psi\phi_i}\right)\right)E_i^{n+1}$$
$$-\left(\psi\left(\alpha_{i+1/2}-\zeta_{i+1/2}v_{x,i+1/2}^n-\frac{\omega_{x,i}v_{x,i}^n}{1+\psi\phi_i}\right)\right)E_{i+1}^{n+1}$$
$$-\left(\psi\left(\alpha_{i-1/2}+\zeta_{i-1/2}v_{x,i-1/2}^n+\frac{\omega_{x,i}v_{x,i}^n}{1+\psi\phi_i}\right)\right)E_{i-1}^{n+1}$$
$$-\left(\psi\left(\alpha_{i-1/2}+\zeta_{i-1/2}v_{x,i-1/2}^n+\frac{\omega_{x,i}v_{x,i}^n}{1+\psi\phi_i}\right)\right)E_{i-1}^{n+1}$$
$$=\left(1-\bar{\psi}\left(\alpha_{i+1/2}+\alpha_{i-1/2}+\zeta_{i+1/2}v_{x,i+1/2}^n-\zeta_{i1/2}v_{x,i-1/2}^n+\frac{\epsilon_i\xi_i}{1+\psi\phi_i}\right)\right)E_i^n$$

2D etc...

For 2D or 3D we have more connections to add to the matrix elements but it is very straight forward... There will be additional terms for each dimension - and the velocity components (v,,x,,) will change, but everything else stays the same.

Initial solution vector For the initial solution vector, we can just use Edot from the parent update (or last time step if we are on the coarse grid) to guess E $E_i^{n+1} = E_i^n + \dot{E}_i^n \Delta t$

2.8.1 Coarse-Fine Boundaries

Since we are doing our implicit solves first, we can use time interpolated solutions for the implicit solve for non-refined ghost zones. To do this we just need Edot. The opacities etc... in the ghost zones can be obtained from the hydro terms.

And the radiative implicit heating in coarse ghost cells can be done along with the initial solution vector so they are available for the hydro update.

2.8.2 Physical Boundary Conditions

For each boundary type we need to specify $E_g^n, E_g^{n+1}, \kappa_g,$ and v_g in terms of other quantities (i.e. including $E^n + 1^i$). The v,,g,, and kappa,,g,, terms should come from the hydro boundary conditions so we just need an equation for $E^{n'}, g,,$

Open (Free streaming) boundaries We would like the radiation to leave at the free streaming limit. But we could have to modify the opacity in the ghost zone to keep the radiation energy positive

$\frac{c\lambda}{\kappa_{0R}} \nabla E = \mathbf{F} = cE\mathbf{n} = \frac{c\lambda_g}{\kappa_g} \frac{(E-E_g)}{\Delta x}$

Clearly if we set $E_g = 0$ and $\kappa_g << \frac{1}{\Delta x}$ we should get $\lambda = \kappa_g \Delta x$ and a free streaming flux of $cE\mathbf{n}$ which also corresponds to an $\alpha = c\frac{\Delta t}{\Delta x}$

Constant Slope Boundary $E_g = 2E_i - E_{i+1}$

Periodic Boundary This is the same as internal zones - it just maps the neighbor cell to be across the domain. Hypre has built in functionality for this under for the Struct Interface

User defined radiation field/Coarse Fine boundary

This will be the boundary at internal coarse-fine boundaries, but could also be used at the physical boundary if the radiation energy were specified.

Reflecting/ZeroSlope Boundary

$E_g = E_i$

Constant radiative flux

$E_g = E_i - F_0 \frac{\kappa_g \Delta x}{c\lambda_g}$