# Lecture, 23 September 1999

## 7.1 On-axis aberrations of conic section mirrors: spherical aberration

To illustrate why the spot diagrams we generated above have the size and shape they do, we will consider a couple of simple examples. First, consider a parallel bundle of rays incident parallel to the axis of a mirror whose surface is a conic section surface of revolution. The ray encountering the mirror at point \( y, z \) reflects at angle \( \theta \), as shown in Figure 7.1. How does the “focal length” \( f \) depend upon the distance from the optical axis \( y \)?

Note that for \textit{paraxial} rays (those with small enough values of \( y \)), we’ve shown before that \( f = 1/2\kappa \), where \( \kappa \) is the curvature of the surface at the apex. Now we want to consider \textit{marginal} rays as well.

![Figure 7.1: geometry for calculation of on-axis aberrations.](image)

From Figure 7.1, we have

\[
z_0 = \frac{y}{\tan 2\theta} = \frac{y}{2\tan \theta} = \frac{y}{2} \left( \frac{1}{\tan \theta} - \tan \theta \right) . \tag{7.1}
\]

The angle \( \theta \) is of course related to the local slope of the mirror:

\[
\tan \theta = \frac{dz}{dy} . \tag{7.2}
\]

In calculating \( dz/dy \), we will find it convenient to use Equation 3.13 for the conic-section apex region:

\[
y^2 - \frac{2}{\kappa} z + (1 - e^2) z^2 = 0 \quad , \tag{7.3}
\]
where \( \kappa \) and \( \varepsilon \) are, as usual, the apex curvature and eccentricity. Upon differentiation with respect to \( y \), we obtain

\[
2y - 2\frac{dz}{\kappa\,dy} + 2(1 - \varepsilon^2)z\frac{dz}{dy} = 0, \tag{7.4}
\]

or

\[
\frac{dz}{dy} = \frac{\kappa y}{1 - \kappa(1 - \varepsilon^2)z}. \tag{7.5}
\]

We can use this new expression in place of \( \tan \theta \) in Equation 7.1:

\[
z_0 = \frac{y}{2} \left[ 1 - \kappa(1 - \varepsilon^2)z \right], \tag{7.6}
\]

Now we can write

\[
f = z + z_0 = z + \frac{1 - \kappa(1 - \varepsilon^2)z}{2\kappa} - \frac{\kappa y^2}{2(1 - \kappa(1 - \varepsilon^2)z)} \tag{7.7}
\]

For \( \varepsilon = 1 \) (paraboloids), \( f = 1/2\kappa \) — the focal length is independent of \( y \), so there are no aberrations, as we’ve noted twice before. For all other figures, however, the location of the intersection with the optical axis is different for rays incident on different places on the mirror, so there is no point image of the distant point object. This particular blurring mechanism is called spherical aberration.

To see explicitly the dependence of \( f \) on \( y \) we need to eliminate \( z \) from Equation 7.7. We can invoke the quadratic formula on Equation 7.3 to obtain a suitable substitute:

\[
z = \frac{2 \pm \sqrt{4 + 4y^2(1 - \varepsilon^2)}}{2(1 - \varepsilon^2)} = 1 - \frac{1}{\kappa} \sqrt{\frac{1 - \varepsilon^2 - y^2}{1 - \varepsilon^2}} \tag{7.8}
\]

(Note that the “minus” root is chosen; this is the point on the conic section nearest the apex, and results in Equation 3.14.) If \( y^2 << 1/\kappa^2 \), as we will now assume, we can use the binomial theorem to expand the square root: if \( |x| << 1 \) and \( s \) is any real number, then

\[
(1 + x)^s = \sum_{n=0}^{\infty} \frac{s!}{n!(s-n)!} x^n = 1 + \frac{s}{1!} x + \frac{s(s-1)}{2!} x^2 + \frac{s(s-1)(s-2)}{3!} x^3 + \ldots \tag{7.9}
\]

Since the terms get smaller and smaller we can ignore them after a few, to good approximation. In our case, \( x = \kappa^2 y^2(1 - \varepsilon^2) \) and \( s = \frac{1}{2} \), so
\[ z = \frac{1}{\kappa(1-\epsilon^2)} \left[ 1 - \left( 1 - \frac{1}{2} \kappa^2 y^2 (1-\epsilon^2)^2 \right) - \frac{1}{8} \kappa^2 y^2 (1-\epsilon^2)^2 - \ldots \right] \]

\[ = \frac{\kappa y^2}{2} \left[ 1 + \frac{(1-\epsilon^2)\kappa^2 y^2}{4} + \ldots \right] . \]  

(7.10)

Let’s ignore terms higher than fourth order in \( y \) (those represented by \( \ldots \)), and use this for \( z \) in the Equation 7.7:

\[ f \equiv \frac{1}{2\kappa} + \frac{(1+\epsilon^2)\kappa y^2}{2} + \frac{(1-\epsilon^2)\kappa^3 y^4}{8} \ldots \]  

\[ = \frac{1}{2\kappa} + \frac{(1+\epsilon^2)\kappa y^2}{4} + \frac{(1+\epsilon^2)(1-\epsilon^2)\kappa^3 y^4}{16} \ldots \]  

\[ - \frac{\kappa y^2}{2} \left[ 2 - \frac{(1-\epsilon^2)\kappa^2 y^2}{2} (1 + \frac{(1-\epsilon^2)\kappa^2 y^2}{4}) \right] . \]  

(7.11)

Once again, if \( \kappa^2 y^2 \ll 1 \), then the term in curly brackets (\( \ldots \)) is small and we can use the binomial theorem again, this time with \( x = \ldots \) and \( s = -1 \). To the same order in \( y \) (fourth) as before, then, we get

\[ f \equiv \frac{1}{2\kappa} + \frac{(1+\epsilon^2)\kappa y^2}{4} + \frac{(1+\epsilon^2)(1-\epsilon^2)\kappa^3 y^4}{16} - \frac{\kappa y^2}{2} \left[ 2 - \frac{(1-\epsilon^2)\kappa^2 y^2}{2} (1 + \frac{(1-\epsilon^2)\kappa^2 y^2}{4}) \right] \]  

(7.12)

Thus, for \( \epsilon \neq 1 \), marginal rays miss the paraxial focus, at \( z = f_{\text{paraxial}} = 1 / 2\kappa \), by an amount

\[ \Delta f = f - f_{\text{paraxial}} - \frac{(1-\epsilon^2)\kappa y^2}{4} - \frac{(3-\epsilon^2)(1-\epsilon^2)\kappa^3 y^4}{16} . \]  

(7.13)

Note that this makes \( \Delta f < 0 \) for spheres and ellipsoids: reflected marginal rays meet the optical axis closer to the apex than the paraxial rays do. The opposite is the case for hyperboloids.

To compare these effects to a spot diagram like those one can generate with RayTrace, it is handy to have an expression for the intersection of reflected rays with the paraxial focal plane. The relevant geometry is shown (for \( \epsilon < 1 \)) in Figure 7.2. Call the distance from the intersection to the center of the paraxial focal plane \( TSA \), for transverse spherical aberration, and calculate this distance in terms of \( y \), \( \epsilon \) and \( \kappa \). From the two similar triangles in Figure 7.2, we have
We can use Equations 7.10 and 7.12 to eliminate \( f-z \):

\[
\frac{TSA}{\Delta f} = \frac{y}{z_0} = \frac{y}{f-z} .
\]  

(7.14)

or, by use of the binomial theorem,

\[
\frac{1}{f-z} = 2\kappa \left[ 1 + \frac{(3-\varepsilon^2)\kappa^2 y^2}{2} + \frac{(5-\varepsilon^2)(1-\varepsilon^2)\kappa^4 y^4}{8} \right] ,
\]

(7.16)

whence

\[
TSA = \frac{y\Delta f}{f-z} = 2\kappa y \left[ \frac{(1-\varepsilon^2)\kappa y^2}{4} - \frac{(3-\varepsilon^2)(1-\varepsilon^2)\kappa^3 y^4}{16} \right] + \frac{(3-\varepsilon^2)(1-\varepsilon^2)\kappa^4 y^4}{8} .
\]

(7.17)

Let us henceforth ignore terms in this expression of higher order than fifth in \( y \); this leaves us with

\[
TSA = -\frac{(1-\varepsilon^2)\kappa^2 y^3}{2} - \frac{(3-\varepsilon^2)(1-\varepsilon^2)\kappa^4 y^5}{8} - \frac{(3-\varepsilon^2)(1-\varepsilon^2)\kappa^4 y^5}{4} .
\]

(7.18)
where the names $TSA_3$ and $TSA_5$ are given in to the two leading terms, after the order of $y$ they carry. $TSA_5$ is pretty small compared to $TSA_3$:

$$\frac{TSA_5}{TSA_3} = \frac{3(3-\varepsilon^2)\kappa^2 y^2}{4} \ll 1$$

(7.19)

so it’s usually sufficient to keep just the third-order term.

The sign of $TSA$ changes when the sign of the apex curvature $\kappa$ conic constant $1-\varepsilon^2$ does. For positive curvature (concave mirrors),

<table>
<thead>
<tr>
<th>Figure</th>
<th>$1-\varepsilon^2$</th>
<th>$TSA$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>1</td>
<td>&lt;0</td>
</tr>
<tr>
<td>Ellipsoid</td>
<td>&gt;0</td>
<td>&lt;0</td>
</tr>
<tr>
<td>Paraboloid</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Hyperboloid</td>
<td>&lt;0</td>
<td>&gt;0</td>
</tr>
</tbody>
</table>

This suggests the use of two mirrors – say, an ellipsoid and a hyperboloid, or one convex and one concave hyperboloid – to cancel each other’s spherical aberration.

### 7.2 Spherical aberration in refraction

Consider next the case of refraction by a single, conic-section dielectric surface. For simplicity we’ll take the surface to be convex from the point of view of the on-axis, parallel, incident rays, and to have curvature $\kappa$ and eccentricity $\varepsilon$. Suppose the dielectrics have index 1 (vacuum) on the incident side and $n$ on the other side. The situation is shown in Figure 7.3. What is the magnitude of spherical aberration now? Is there a surface for which the SA is zero, as it is for paraboloids in reflection?

To answer these questions is straightforward, and only slightly more complicated algebraically than the case of the conic-section mirrors above. The use of Snell’s Law instead of the mirror-reflection rule gives us terms with sines and arcsines, tangents and arctangents, that didn’t appear in the last example, but other than that the procedure is the same as before. We obtain, to fifth order in $y$,

$$TSA = -\frac{1}{2} \frac{(1-n^2\varepsilon^2)}{n^2} \kappa^2 y^3 - \frac{3}{8} \frac{n^4 (\varepsilon^4 - \varepsilon^2) + n^3 \varepsilon^2 + n^2(1-2\varepsilon^2) - n + 1}{n^4} \kappa^4 y^5 .$$

(7.20)

Several remarks are in order concerning this result.

- Substitution of $n = -1$ yields the result for the concave mirror (Equation 7.18), which is nice but a bit accidental. (So does $n = 1$, which you may find confusing; if so just note that this gives $\theta = \theta'$, so that the ray never intersects the axis.)

- A paraboloid surface does not lead to zero spherical aberration in this case, as it does for concave mirrors. Instead the zero-SA surface is an ellipsoid, with $\varepsilon = 1 / n$.

- $TSA$ decreases rapidly as $n$ increases; thus SA may be reduced simply by using a higher-index dielectric. In turn this leads to more light reflected from the surface and less power actually
transmitted, a loss that may be ameliorated by applying antireflection coatings to the dielectric surface. (We will discuss AR coatings later on in this course.)

Homework problem 3.1. Derive Equation 7.20. In the process you will find useful the following power-series representations:

\[
\begin{align*}
\sin \theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots \\
\sin^{-1} x &= x - \frac{x^3}{3} + \frac{3x^5}{40} + \ldots \\
\tan \theta &= \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \ldots \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots
\end{align*}
\]  

(7.21)

7.3 Angular aberrations and reference surfaces

Another useful way to cast the result 7.18 is to define the angular spherical aberration. Consider an on-axis ray incident at \(y\) on two slightly different mirrors: a paraboloid (which of course lacks spherical aberration) and another type of conic section mirror with the same paraxial focal length \(1/2\kappa\), as shown (for a negative-TSA surface) in Figure 7.4. The ray is incident on the paraboloid at an angle \(\theta_p\) would reflect from the paraboloid through the focus, making an angle \(2\theta_p\) with the line along its original direction. With the other conic-section mirror the ray would be reflected similarly at an angle \(2\theta\), not through the focus. The angular spherical aberration is defined from these angles as

\[
ASA = 2\theta_p - 2\theta.
\]  

(7.22)
If these angles were small, we could replace their radian measures by their tangents: \( \theta \equiv \tan \theta = dz / dy \), \( \theta_p \equiv \tan \theta_p = dz_p / dy \). Thus

\[
\text{ASA} \equiv 2 \frac{dz_p}{dy} - 2 \frac{dz}{dy} = \frac{d}{dy} (2\Delta z),
\tag{7.23}
\]

where \( \Delta z = z_p(y) - z(y) \) is the distance between the intersections of the ray with the two surfaces, as shown in Figure 7.4. Now, \( z_p = \kappa y^2 / 2 \); combining this with Equation 7.10, we have, to fourth order in \( y \),

\[
\Delta z \equiv -\frac{(1 - \epsilon^2)\kappa^3 y^4}{8}.
\tag{7.24}
\]

Therefore, to third order in \( y \),

\[
\text{ASA} \equiv \text{ASA}_3 = \frac{d}{dy} (2\Delta z) = -(1 - \epsilon^2)\kappa^3 y^3.
\tag{7.25}
\]

With a glance back to Equation 7.18 we see that

\[
\text{TSA}_3 = \frac{1}{2\kappa} \text{ASA}_3.
\tag{7.26}
\]

This is an example of a generally useful formulation for the third-order aberrations - to find the aberrations produced by a given reflecting surface, construct a surface with no aberrations, calculate \( \Delta z \), and use

Figure 7.4: comparison of reflection of an on-axis ray by two mirrors of identical paraxial focal length, with (broken lines) and without (solid lines) spherical aberration.
\[ AA = \frac{d}{dy} (2\Delta z) \]
\[ TA = \frac{1}{2\kappa} \Delta A \]