Today in Physics 217: the divergence and curl theorems

- Flux and divergence: proof of the divergence theorem, à la Purcell.
- Circulation and curl: proof of Stokes’ theorem, also following Purcell.

See Purcell, chapter 2, for more information.

\[
\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \oint_{C_1} \mathbf{v} \cdot d\mathbf{l}_1 + \oint_{C_2} \mathbf{v} \cdot d\mathbf{l}_2
\]
The divergence theorem

Consider a vector function $\mathbf{v}$ that exists everywhere in space, and a specific volume $V$, bounded by surface $S$. The flux of $\mathbf{v}$ through $S$ is given by

$$\Phi = \oint_S \mathbf{v} \cdot d\mathbf{a}$$

where the infinitesimal area element vectors on $S$, $d\mathbf{a}$, point outward and are perpendicular to $S$.

Now consider dividing $V$ in two:
The divergence theorem (continued)

The two new volumes have more total surface area on their boundaries than they used to, by twice the area of the dividing surface $D$. But because the area elements point outward from each volume, the flux through $D$ from $V_1$ is equal and opposite to that from $V_2$, so the total flux through all the surfaces is the same as it was before:

$$\Phi = \oint_S v \cdot da = \oint_{S_1} v \cdot da_1 + \oint_{S_2} v \cdot da_2$$
The divergence theorem (continued)

In fact, if we were to subdivide $V$ into $N \gg 1$ cells, the flux through all the dividing surfaces would cancel out in the sum, though the total surface area would be much larger than originally:

$$\Phi = \oint_{S} \mathbf{v} \cdot d\mathbf{a} = \sum_{i=1}^{N} \oint_{S_{i}} \mathbf{v} \cdot d\mathbf{a}_{i}$$

We now claim that

$$\lim_{V_{i} \to 0} \left( \frac{\oint_{S_{i}} \mathbf{v} \cdot d\mathbf{a}_{i}}{V_{i}} \right) = \nabla \cdot \mathbf{v} .$$

$S_{i}$ bounds $V_{i}$. 
Note:

We’re about to examine the variation of a function $v$ in a very small volume near the point $x,y,z$. Recall that if the volume is small enough that $v$ changes very little within it, then the value of $v$ at $x+\delta x, y+\delta y, z+\delta z$ would be

$$v(x+\delta x, y+\delta y, z+\delta z) \approx v(x,y,z) + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z$$

One can see this from the definition of the derivative, or by expanding $v$ in a Taylor series about the point $x,y,z$ and neglecting all the terms of higher order than those with first derivatives – OK, because they have successively higher powers of the intervals $\delta x, \delta y, \delta z$ and are thus much smaller than the zeroth and first-derivative terms.
The divergence theorem (continued)

To justify the claim, consider a small box within \( V \), lying with one corner at \( r_i = (x, y, z) \) with volume \( \Delta x \Delta y \Delta z \), as one of the \( V_i \). Its contribution to the flux is

\[
\int \mathbf{v} \cdot d\mathbf{a}_i = \sum_{\text{sides}} \mathbf{v} \cdot \Delta a_i
\]

Look at the top and bottom sides first. Their area vectors are equal and opposite and lie along \( z \), and the top lies \( \Delta z \) above the bottom. If the box is indeed small, the value of the \( z \) component of \( \mathbf{v} \) in the center of the top face is approximately

\[
v_{z, \text{top}} \approx v_z(x, y, z) + \frac{\partial v_z}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_z}{\partial y} \frac{\Delta y}{2} + \frac{\partial v_z}{\partial z} \Delta z
\]
The divergence theorem (continued)

The value of $v_z$ in the center of the bottom face is, on the other hand,

$$v_{z, \text{bot.}} \approx v_z(x, y, z) + \frac{\partial v_z}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_z}{\partial y} \frac{\Delta y}{2}$$

Take the value at the center of each face to serve as the average of $v_z$ on the face. Then the flux contributed by the top and bottom is

$$v_{z, \text{top}} \Delta a_{\text{top}} + v_{z, \text{bot.}} \Delta a_{\text{bot.}} \approx \left( v_z(x, y, z) + \frac{\partial v_z}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_z}{\partial y} \frac{\Delta y}{2} + \frac{\partial v_z}{\partial z} \Delta z \right) (\Delta x \Delta y)$$

$$+ \left( v_z(x, y, z) + \frac{\partial v_z}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_z}{\partial y} \frac{\Delta y}{2} \right) (-\Delta x \Delta y)$$

$$= \frac{\partial v_z}{\partial z} \Delta x \Delta y \Delta z$$
The divergence theorem (continued)

Similarly, the front and back, and right and left, pairs contribute

\[ v_{x, \text{fr.}} \Delta a_{\text{fr.}} + v_{x, \text{ba.}} \Delta a_{\text{ba.}} \approx \left( v_x (x, y, z) + \frac{\partial v_x}{\partial z} \frac{\Delta z}{2} + \frac{\partial v_x}{\partial y} \frac{\Delta y}{2} + \frac{\partial v_x}{\partial x} \Delta x \right) (\Delta y \Delta z) \]

\[ + \left( v_x (x, y, z) + \frac{\partial v_x}{\partial z} \Delta z + \frac{\partial v_x}{\partial y} \Delta y \right) (-\Delta y \Delta z) \]

\[ = \frac{\partial v_x}{\partial x} \Delta x \Delta y \Delta z \]

\[ v_{y, \text{R}} \Delta a_{\text{R}} + v_{y, \text{L}} \Delta a_{\text{L}} \approx \left( v_y (x, y, z) + \frac{\partial v_y}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_y}{\partial z} \frac{\Delta z}{2} + \frac{\partial v_y}{\partial y} \Delta y \right) (\Delta x \Delta z) \]

\[ + \left( v_y (x, y, z) + \frac{\partial v_y}{\partial x} \Delta x + \frac{\partial v_y}{\partial z} \Delta z \right) (-\Delta x \Delta z) \]

\[ = \frac{\partial v_y}{\partial y} \Delta x \Delta y \Delta z \]
The divergence theorem (continued)

Thus
\[ \int v \cdot da_i \approx \sum_{\text{sides}} v \cdot \Delta a_i = \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \Delta x \Delta y \Delta z \]
\[ = \nabla \cdot v (r_i) V_i , \]
and
\[ \lim_{V_i \to 0} \left( \int_{S_i} v \cdot da_i \right) V_i = \nabla \cdot v , \text{ just as we claimed.} \]

Back to the flux:

\[ \Phi = \oint_S v \cdot da = \sum_{i=1}^{N} \oint_{S_i} v \cdot da_i \approx \sum_{i=1}^{N} V_i \left( \int_{S_i} v \cdot da_i \right) V_i = \sum_{i=1}^{N} V_i (\nabla \cdot v)_i \rightarrow \int_{V} \nabla \cdot v d\tau , \text{ or} \]
\[ \oint_S v \cdot da = \int_{V} \nabla \cdot v d\tau \quad \text{Divergence theorem} \]
The divergence theorem (continued)

The meaning of the divergence theorem is illustrated by applying it to fluid flow, with $\mathbf{v}$ as the fluid velocity:

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{v} \, d\mathbf{r}$$

Total flow through \text{bounding surface} = \text{Difference of numbers of faucets and drains within volume}

The divergence theorem will eventually lead us to Gauss’ law.
Stokes’ theorem

Consider a vector function \( \mathbf{v} \) that exists everywhere in space, and a specific surface \( S \) (not necessarily planar), bounded by curve \( C \). The circulation of \( \mathbf{v} \) around \( C \) is given by

\[
\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l}
\]

where the infinitesimal line element vectors on \( C \), \( d\mathbf{l} \), point parallel to \( C \) in the right-hand-rule sense (as in \( d\mathbf{a} = d\mathbf{l}_a \times d\mathbf{l}_b \)) relative to the area element vectors on \( S \).
Stokes’ theorem (continued)

Now divide the area in two. The two new areas have more total circumference than they used to, by twice the length of the dividing line $D$. But because the line elements follow each curve counterclockwise, the line integral along $B$ around $S_1$ is equal and opposite to that from $S_2$, so the sum of the two line integrals is the same as the original:

$$\Gamma = \oint_C v \cdot dl = \oint_{C_1} v \cdot dl_1 + \oint_{C_2} v \cdot dl_2$$
Stokes’ theorem (continued)

In fact, if we were to subdivide $S$ into $N$ ($>>1$) cells, the line integrals along all the dividing lines would cancel out in the sum, though the sum of circumferences would be much larger than originally:

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \sum_{i=1}^{N} \oint_{C_i} \mathbf{v} \cdot d\mathbf{l}_i$$

I bet you can guess what we’ll claim now:

$$\lim_{a_i \to 0} \left( \frac{\oint_{C_i} \mathbf{v} \cdot d\mathbf{l}_i}{a_i} \right) = \hat{n} \cdot (\nabla \times \mathbf{v}) .$$

$C_i$ bounds $S_i$; area of $S_i$ is $a_i = a_i \hat{n}_i$
Stokes’ theorem (continued)

To justify the claim, consider a small box within $S$, lying with one corner at $r_i = (x, y, z)$ with area $a_i = \hat{z}\Delta x\Delta y$, as one of the $S_i$. Its contribution to the total circulation is

$$\oint_{C_i} \mathbf{v} \cdot d\mathbf{l} = \sum_{\text{sides}} \mathbf{v} \cdot \Delta \mathbf{l}_i$$

Look at the top and bottom sides first. Their line element vectors are equal and opposite and lie along $x$, and the top lies $\Delta y$ above the bottom. If the box is indeed small, the value of the $x$ component of $\mathbf{v}$ in the center of the top side is approximately

$$v_{x, T} \approx v_x(x, y, z) + \frac{\partial v_x}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_x}{\partial y} \Delta y$$
Stokes’ theorem (continued)

The value of \( v_x \) in the center of the bottom side is, on the other hand,

\[
v_{x, B} \approx v_x(x, y, z) + \frac{\partial v_x}{\partial x} \frac{\Delta x}{2}
\]

Take the value at the center of each side to serve as the average of \( v_x \) on the side. Then the contribution to the circulation by the top and bottom is

\[
v_{x, T} \Delta l_T + v_{x, B} \Delta l_B \approx \left( v_x(x, y, z) + \frac{\partial v_x}{\partial x} \frac{\Delta x}{2} + \frac{\partial v_x}{\partial y} \Delta y \right)(-\Delta x)
\]

\[
+ \left( v_x(x, y, z) + \frac{\partial v_x}{\partial x} \frac{\Delta x}{2} \right)(\Delta x) = -\frac{\partial v_x}{\partial y} \Delta x \Delta y
\]
Stokes’ theorem (continued)

Similarly, the left and right sides give

\[ v_y, R \Delta l_R + v_y, L \Delta l_L \equiv \left( v_y(x, y, z) + \frac{\partial v_y}{\partial y} \frac{\Delta y}{2} + \frac{\partial v_y}{\partial x} \Delta x \right)(\Delta y) \]

\[ + \left( v_y(x, y, z) + \frac{\partial v_y}{\partial y} \frac{\Delta y}{2} \right)(-\Delta y) = \frac{\partial v_y}{\partial x} \Delta x \Delta y \]

So, noting that \( a_i = \hat{z} \Delta x \Delta y \), we have for all four sides together

\[ \oint v \cdot dl = \sum_{\text{sides}} v \cdot \Delta l_i \equiv \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \Delta x \Delta y = (\nabla \times v) \cdot a_i \]

Rest assured that we would get the same result with the area vector in any direction; thus our claim is justified.
Stokes’ theorem (continued)

Thus

\[ \Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \sum_{i=1}^{N} \oint_{C_i} \mathbf{v} \cdot d\mathbf{l}_i = \sum_{i=1}^{N} (\nabla \times \mathbf{v}) \cdot \mathbf{a}_i \xrightarrow{N \to \infty} \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} \, , \]

or \[ \oint_C \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} \, . \]

\textbf{Stokes’ theorem}