Today in Physics 217: the divergence and curl theorems

- Flux and divergence: proof of the divergence theorem, à la Purcell.
- Circulation and curl: proof of Stokes’ theorem, also following Purcell.

See Purcell, chapter 2, for more information.

The divergence theorem

Consider a vector function \( \mathbf{v} \) that exists everywhere in space, and a specific volume \( V \), bounded by surface \( S \). The flux of \( \mathbf{v} \) through \( S \) is given by

\[
\Phi = \oint_S \mathbf{v} \cdot d\mathbf{a}
\]

where the infinitesimal area element vectors on \( S \), \( d\mathbf{a} \), point outward and are perpendicular to \( S \).

Now consider dividing \( V \) in two:

The divergence theorem (continued)

The two new volumes have more total surface area on their boundaries than they used to, by twice the area of the dividing surface \( D \). But because the area elements point outward from each volume, the flux through \( D \) from \( V_1 \) is equal and opposite to that from \( V_2 \), so the total flux through all the surfaces is the same as it was before:

\[
\Phi = \oint_{S_1} \mathbf{v} \cdot d\mathbf{a}_1 + \oint_{S_2} \mathbf{v} \cdot d\mathbf{a}_2
\]
The divergence theorem (continued)

In fact, if we were to subdivide \( V \) into \( N \) cells, the flux through all the dividing surfaces would cancel out in the sum, though the total surface area would be much larger than originally:

\[
\Phi = \oint_S \mathbf{v} \cdot d\mathbf{a} = \sum_{i=1}^N \oint_{S_i} \mathbf{v} \cdot d\mathbf{a}_i
\]

We now claim that

\[
\lim_{N \to \infty} \left( \frac{\oint_{S_i} \mathbf{v} \cdot d\mathbf{a}_i}{V_i} \right) = \nabla \cdot \mathbf{v} .
\]

Note:

We’re about to examine the variation of a function \( v \) in a very small volume near the point \( x, y, z \). Recall that if the volume is small enough that \( v \) changes very little within it, then the value of \( v \) at \( x+\delta x, y+\delta y, z+\delta z \) would be

\[
v(x+\delta x, y+\delta y, z+\delta z) \approx v(x, y, z) + \left( \frac{\partial v}{\partial x} \right)_x \delta x + \left( \frac{\partial v}{\partial y} \right)_y \delta y + \left( \frac{\partial v}{\partial z} \right)_z \delta z
\]

One can see this from the definition of the derivative, or by expanding \( v \) in a Taylor series about the point \( x, y, z \) and neglecting all the terms of higher order than those with first derivatives – OK, because they have successively higher powers of the intervals \( \delta x, \delta y, \delta z \) and are thus much smaller than the zeroth and first-derivative terms.

The divergence theorem (continued)

To justify the claim, consider a small box within \( V \), lying with one corner at \( r = (x, y, z) \) with volume \( \Delta x \Delta y \Delta z \), as one of the \( V_i \). Its contribution to the flux is

\[
\oint_{S_i} \mathbf{v} \cdot d\mathbf{a}_i = \int_{S_i} \mathbf{v} \cdot d\mathbf{a}_i
\]

Look at the top and bottom sides first. Their area vectors are equal and opposite and lie along \( z \), and the top lies \( \Delta z \) above the bottom. If the box is indeed small, the value of the \( z \) component of \( \mathbf{v} \) in the center of the top face is approximately

\[
v_{z, \text{top}} \approx v_z(x, y, z + \frac{\Delta z}{2})
\]
The divergence theorem (continued)

The value of \( v_z \) in the center of the bottom face is, on the other hand, 
\[
v_z(\text{bot}) = \frac{1}{2} \left( \frac{\partial v_x}{\partial x} \bigg|_{x=x'} + \frac{\partial v_y}{\partial y} \bigg|_{y=y'} \right) \Delta x \Delta y \\
\]
Take the value at the center of each face to serve as the average of \( v_z \) on the face. Then the flux contributed by the top and bottom is 
\[
\Phi_{\text{top}} = \nabla \cdot \mathbf{v} \Delta \mathbf{A} = \nabla \cdot \mathbf{v} \Delta x \Delta y \\
\Phi_{\text{bot}} = \nabla \cdot \mathbf{v} \Delta \mathbf{A} = -\nabla \cdot \mathbf{v} \Delta x \Delta y
\]

Similarly, the front and back, and right and left, pairs contribute
\[
v_{x, \text{fr}} \Delta \mathbf{A}_{x} = \nabla \cdot \mathbf{v} \Delta x \Delta y \\
v_{x, \text{ba}} \Delta \mathbf{A}_{x} = -\nabla \cdot \mathbf{v} \Delta x \Delta y \\
v_{y, \text{fr}} \Delta \mathbf{A}_{y} = \nabla \cdot \mathbf{v} \Delta x \Delta y \\
v_{y, \text{ba}} \Delta \mathbf{A}_{y} = -\nabla \cdot \mathbf{v} \Delta x \Delta y
\]

Thus
\[
\int_{S} \mathbf{v} \cdot d\mathbf{A} = \sum_{i} \nabla \cdot \mathbf{v} \Delta \mathbf{A} = \nabla \cdot \mathbf{v} \int_{V} \Delta x \Delta y \\
-\mathbf{v} \cdot \nabla (\Phi) + \nabla \cdot \mathbf{v} \rightarrow -\mathbf{v} \cdot \nabla (\Phi) + \nabla \cdot \mathbf{v} = \Phi_{\text{total}}
\]
and
\[
\lim_{V_{i} \rightarrow 0} \frac{1}{V_{i}} \int_{S} \mathbf{v} \cdot d\mathbf{A} = -\mathbf{v} \cdot \nabla (\Phi) + \nabla \cdot \mathbf{v} = \Phi_{\text{total}}
\]

Back to the flux:
\[
\Phi_{\text{total}} = \int_{S} \mathbf{v} \cdot d\mathbf{A} = \sum_{i} \int_{V_{i}} \mathbf{v} \cdot d\mathbf{A} = \sum_{i} \int_{V_{i}} \left( \frac{1}{2} \left( \frac{\partial v_{x}}{\partial x} \bigg|_{x=x'} + \frac{\partial v_{y}}{\partial y} \bigg|_{y=y'} \right) \Delta x \Delta y \right)
\]

Divergence theorem
The divergence theorem (continued)

The meaning of the divergence theorem is illustrated by applying it to fluid flow, with \( \mathbf{v} \) as the fluid velocity:

\[
\oint\mathbf{v} \cdot d\mathbf{a} = \iiint (\nabla \cdot \mathbf{v}) \, dV
\]

Total flow through bounding surface \( \oint \mathbf{v} \cdot d\mathbf{a} \) = Difference of numbers of faucets and drains within volume

The divergence theorem will eventually lead us to Gauss’ law.

Stokes’ theorem

Consider a vector function \( \mathbf{v} \) that exists everywhere in space, and a specific surface \( S \) (not necessarily planar), bounded by curve \( C \). The circulation of \( \mathbf{v} \) around \( C \) is given by

\[
\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l}
\]

where the infinitesimal line element vectors on \( C \), \( d\mathbf{l} \), point parallel to \( C \) in the right-hand-rule sense (as in \( da = dl_x \times dl_y \)) relative to the area element vectors on \( S \).

Stokes’ theorem (continued)

Now divide the area in two. The two new areas have more total circumference than they used to, by twice the length of the dividing line \( D \). But because the line elements follow each curve counterclockwise, the line integral along \( B \) around \( S_1 \) is equal and opposite to that from \( S_2 \), so the sum of the two line integrals is the same as the original:

\[
\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l} = \oint_{C_1} \mathbf{v} \cdot d\mathbf{l}_1 + \oint_{C_2} \mathbf{v} \cdot d\mathbf{l}_2
\]
Stokes’ theorem (continued)

In fact, if we were to subdivide $S$ into $N \gg 1$ cells, the line integrals along all the dividing lines would cancel out in the sum, though the sum of circumferences would be much larger than originally:

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \sum_{i=1}^{N} \oint_{C_i} \mathbf{v} \cdot d\mathbf{l}$$

I bet you can guess what we’ll claim now:

$$\lim_{N \to \infty} \left( \frac{\oint \mathbf{v} \cdot d\mathbf{l}}{\text{area of } S_i} \right) = \mathbf{\hat{n}} \cdot (\nabla \times \mathbf{v}) \quad \text{for } C_i \text{ bounds } S_i; \text{ area of } S_i$$

Stokes’ theorem (continued)

To justify the claim, consider a small box within $S$, lying with one corner at $\mathbf{r}_i = (x, y, z)$ with area $A_i = \Delta x \Delta y \Delta z$, as one of the $S_i$. Its contribution to the total circulation is

$$\oint_{C_i} \mathbf{v} \cdot d\mathbf{l} = \sum_{\text{sides}} \oint_{C_i} \mathbf{v} \cdot d\mathbf{l}$$

Look at the top and bottom sides first. Their line element vectors are equal and opposite and lie along $x$, and the top lies $\Delta y$ above the bottom. If the box is indeed small, the value of the $x$ component of $\mathbf{v}$ in the center of the top side is approximately

$$v_x, T \approx v_x(x, y, z) + \left( \frac{\partial v_x}{\partial x} \Delta x \right) + \left( \frac{\partial v_x}{\partial y} \Delta y \right)$$

Stokes’ theorem (continued)

The value of $v_x$ in the center of the bottom side is, on the other hand,

$$v_x, B \approx v_x(x, y, z) + \left( \frac{\partial v_x}{\partial x} \Delta x \right) + \left( \frac{\partial v_x}{\partial y} \Delta y \right)$$

Take the value at the center of each side to serve as the average of $v_x$ on the side. Then the contribution to the circulation by the top and bottom is

$$v_x(T) \Delta l + v_x(B) \Delta l = \left( v_x(x, y, z) + \left( \frac{\partial v_x}{\partial x} \Delta x \right) + \left( \frac{\partial v_x}{\partial y} \Delta y \right) \right) \Delta l$$
Stokes’ theorem (continued)

Similarly, the left and right sides give

\[ \nabla \times \mathbf{A} \cdot d\mathbf{r} = \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \Delta y \Delta x + \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \Delta z \Delta y + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \Delta x \Delta z \]

So, noting that \( A_i = 2 \Delta x \Delta y \), we have for all four sides together

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{N} \mathbf{F} \cdot d\mathbf{r}_i = \sum_{i=1}^{N} \left( \frac{\partial F_y}{\partial x} \Delta x \Delta y - (\nabla \times \mathbf{F}) \cdot \mathbf{n} \right) \]

Rest assured that we would get the same result with the area vector in any direction; thus our claim is justified.

Stokes’ theorem (continued)

Thus

\[ \Gamma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{N} \mathbf{F} \cdot d\mathbf{r}_i = \sum_{i=1}^{N} \left( \nabla \times \mathbf{F} \right) \cdot d\mathbf{a}_i \]

or

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} \quad \text{Stokes’ theorem} \]