Today in Physics 217: begin electrostatics

- Fields and potentials, and the Helmholtz theorem
- The empirical basis of electrostatics
- Coulomb’s Law

At right: the classic hand-to-the-van-de-Graaf experiment.
Vector derivatives, sources, and potentials

Recall that the divergence and curl of vector fields identify the sources of those fields.

- Thus if one knows a field $F$, one can calculate directly its source, from the field’s divergence (if the source is a scalar) or curl (if the source is a vector).

- OR if one knows the source, say a scalar function $D$, or a vector function $C$ for which $\nabla \cdot C = 0$ (just as it is for the curl of $F$), one can work out the field by solving the corresponding differential equation:

$$\nabla \cdot F = D \quad \nabla \times F = C$$

**Question**: do those sorts of differential equations have unique solutions?
Answer: Yes. The solution exists, is unique, and moreover is always of the form

\[ F = -\nabla U + \nabla \times W \]

where

\[ U(r) = \frac{1}{4\pi} \int_\mathcal{V} \frac{1}{|r - r'|} D(r') \, d\tau' \]

Scalar and vector potential

\[ \mathcal{W}(r) = \frac{1}{4\pi} \int_\mathcal{V} \frac{1}{|r - r'|} C(r') \, d\tau' \]

and under the assumptions that, as \( r \to \infty \),

\[ F \to 0, \quad r^2 \nabla \cdot F \to 0, \quad \text{and} \quad r^2 \nabla \times F \to 0 \]

This is called the **Helmholtz theorem** (see Appendix B).
Vector derivatives, sources, and potentials
(continued)

Two immediate consequences of the Helmholtz theorem are particularly relevant in our work this semester:

- **Irrotational fields.** If a vector function is such that $F = -\nabla U$, then all of the following are true:

\[
\nabla \times F = 0.
\]

\[
\int_{a}^{b} F \cdot dl \text{ is independent of path, given } a \text{ and } b.
\]

\[
\oint F \cdot dl = 0 \text{ (thus "irrotational").}
\]

In electrostatics, the electric field $E$ is irrotational, and is the gradient of the (scalar) potential $V$, as we will see soon.
Vector derivatives, sources, and potentials (continued)

- Solenoidal fields. If a vector function is such that
  \( F = \nabla \times W \), then all of the following are true:

  \[ \nabla \cdot F = 0. \]

  \[ \int_S F \cdot da \text{ is independent of surface, given the boundary } C. \]

  \[ \oint_C F \cdot da = 0. \]

  In magnetostatics, the magnetic field \( B \) is solenoidal, and
  is the curl of the magnetic vector potential, \( A \).
Vector derivatives, sources, and potentials
(continued)

Note that although $F$ is a unique solution to the differential equations, the potentials $U$ and $W$ are not, just because the field depends upon the gradient and curl of these potentials, and:

- if a constant is added to $U$, the result has the same gradient, because the gradient of a constant is zero.
- Similarly, if a gradient is added to $W$, the result has the same curl, because the curl of a gradient is zero.

Thus we can add an arbitrary constant to the scalar potential, and the gradient of an arbitrary scalar function to the vector potential, without changing the physics. To make special choices of these “offsets” can make certain problems easier to set up. A set of such choices is called a **gauge**.
Foundation of electrostatics

Matter is composed of compact particles that carry electric charge:

- **Electron**: no spatial structure seen; your basic point object.
- **Proton**: has substructure, but on extremely small scale.

The charge on electron and proton are observed to be equal and opposite, within an accuracy of one part in $10^{20}$.

The electron is said to have negative charge, the proton positive charge.

- The signs we use are a historical accident (and the first permanent mark on science made by an American); the electron could just as well have been defined as the carrier of positive charge.
Foundation of electrostatics (continued)

Five memorable facts about electric charge, determined experimentally:

- **Charge is a scalar quantity.**
- **Conservation:** the algebraic sum of charges (net charge) of an isolated system never changes.
- **Quantization:** the smallest unit of charge observed in nature is that on the proton or electron. All charged bodies have charges that are integer multiples of this quantum of charge.
- **Coulomb’s law:** the force between two charges $q_1$ and $q_2$ is given by
  \[ F_2 = K \frac{q_1 q_2}{r^2} \hat{r} \]
Foundation of electrostatics (continued)

Superposition. The force on a test charge $Q$, located at $r$, by an assembly of point charges $q_i$, is the vector sum of the forces on $Q$ by the individual charges:

$$F(r) = K \frac{Qq_1}{r_1^2} \hat{r}_1 + K \frac{Qq_2}{r_2^2} \hat{r}_2 + \ldots$$

$$= Q \sum_{i=1}^{N} K \frac{q_i}{r_i^2} \hat{r}_i \equiv QE(r)$$

Electric field

Electrostatics is the use of this last expression, or (more often) its consequences, to derive $E$ from a given distribution of $q_i$, or infer a distribution of $q_i$ given $E$. 
Units and dimensions

In Griffiths, as in most undergraduate textbooks, Coulomb’s law and all the rest of the expressions of electromagnetism are written in SI (a.k.a. MKS) units:

$$ F = \frac{1}{4\pi \varepsilon_0} \frac{qQ}{r^2} \hat{r} $$

$$ \varepsilon_0 = 8.85 \times 10^{-12} \text{ coul}^2 \text{ Nt}^{-1} \text{ m}^{-2} $$

In Purcell, as by most professional physicists and astronomers, Gaussian (a.k.a. CGS) units are used:

$$ F = \frac{qQ}{r^2} \hat{r} $$

because the use of CGS units is advantageous in every topic of electromagnetism except electronics. You must learn both systems. I’ll lecture always in CGS, to get you used to it.

Electric charge doesn’t simply have different units in the two systems; it has different dimensions!
Electric charge distributions

The electric charge on a proton is

\[ e_{\text{MKS}} = 1.60217733 \times 10^{-19} \text{ coul}, \]

\[ e_{\text{CGS}} = 4.803206 \times 10^{-10} \text{ esu} \]

That’s so small that in macroscopic systems we can treat charge as a continuous quantity:

\[ E(r) = \int \frac{\hat{r}}{r} dq \]

and speak of charge **density**:

\[ dq = \lambda (r') d\ell' \quad \lambda = \text{charge/unit length for 1-D charges} \]

\[ = \sigma (r') da' \quad \sigma = \text{charge/unit area for 2-D charges} \]

\[ = \rho (r') d\tau' \quad \rho = \text{charge/unit volume for 3-D charges} \]
Examples: calculation of E from charge distributions, using Coulomb’s law

Griffiths, problem 2.2

a. Find the electric field (magnitude and direction) a distance \( z \) above the midpoint between two equal charges \( q \) a distance \( d \) apart. Check that your result is consistent with what you would expect when \( z \gg d \).

\[
\text{Answer: } \quad E = 2qz \left( z^2 + \frac{d^2}{4} \right)^{-3/2} \hat{z}
\]

b. Repeat part a, only this time make the right-hand charge \(-q\) instead of \( +q \).

\[
\text{Answer: } \quad E = qd \left( z^2 + \frac{d^2}{4} \right)^{-3/2} \hat{x}
\]

See also Example 2.1 in Griffiths for a problem with similar geometry.
Griffiths, problem 2.2

\[ \theta \]

\[ x \]

\[ z \]

\[ E_{tot} \]

\[ E_R \]

\[ E_L \]

\[ E_z \]

\[ d/2 \]

\[ q \]

\[ -q \]

\[ E_x \]

\[ E_y \]

\[ P \]
**Examples: calculation of $E$ from charge distributions, using Coulomb’s law (continued)**

**Griffiths, problem 2.5**

Find the electric field at a distance $z$ above the center of a circular loop, of radius $r$, which carries a uniform line charge $\lambda$. Answer:

$$E = qz \left( z^2 + r^2 \right)^{-3/2} \hat{z}$$

$$dq = \lambda rd\phi$$