Today in Physics 217: work and energy in electrostatics

- Last spherical example: potential from a uniformly-charged spherical shell
- Work and potential energy
- Electrostatic potential energy
- Inconsistency?
- Non-superposition of potential energy
Potential from a uniformly-charged spherical shell

Griffiths, example 2.7:
What is the electric potential a distance $z$ away from the center of a spherical shell with radius $R$ and uniform surface charge density $\sigma$?

Please accept my apologies for doing an example that’s worked out in the book. But it goes with the last few examples, and I’d like to present them as a set. Compare especially to problem 2.7, done in class Friday.
Potential from a uniformly-charged spherical shell (continued)

As we saw on Friday,

\[ \mathbf{r}^2 = R^2 + z^2 - 2Rz \cos \theta' \]

so

\[ V(z) = \int \frac{\sigma \, da'}{r} = \sigma \int_0^{2\pi} d\phi' \int_0^\pi \sqrt{R^2 + z^2 - 2Rz \cos \theta'} \, R^2 \sin \theta' \, d\theta' \]

The first integral is trivial: it just comes out to \(2\pi\). For the second, substitute

\[ w = \cos \theta', \quad dw = \sin \theta' \, d\theta', \quad w = 1 \rightarrow -1:\]

so

\[ V(z) = 2\pi \sigma R^2 \int_{-1}^{1} \frac{dw}{\sqrt{R^2 + z^2 - 2Rzw}} \]
Potential from a uniformly-charged spherical shell (continued)

Then, substitute

\[ u = R^2 + z^2 - 2Rzw, \quad du = 2Rzdw, \]

\[ u = R^2 + z^2 + 2Rz \to R^2 + z^2 - 2Rz \]

to get

\[
V(z) = \frac{\pi \sigma R}{z} \int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-1/2} du = \frac{\pi \sigma R}{z} \left[ 2\sqrt{u} \right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} = 2\pi \sigma R \left( \sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz} \right)
\]

Almost at the answer already! But, as before, we must take positive square roots here, so note that

\[ R^2 + z^2 \pm 2Rz = (z \pm R)^2 \]
Potential from a uniformly-charged spherical shell
(continued)

so that we get
\[ V(z) = \frac{2\pi \sigma R}{z} (|z + R| - |z - R|) \]

Two cases: \( z \) larger than, or smaller than, \( R \). (\( P \) outside, inside)

- **Larger (outside):**
  \[ V(z) = \frac{2\pi \sigma R}{z} (z + R - z + R) = \frac{4\pi R^2 \sigma}{z} = \frac{Q}{z} \]

- **Smaller (inside):** means \(|z - R| = R - z\), so
  \[ V(z) = \frac{2\pi \sigma R}{z} (z + R - R + z) = 4\pi R \sigma = \frac{Q}{R} \]

Much easier than Friday’s calculation!

Behaves like a point charge at the sphere’s center.

Constant potential (thus zero field) inside spherical shell.
Work done in motion of a test charge

Move a test charge $Q$ around in the field of a collection of other charges. How much work is done moving it from $a$ to $b$?

The force exerted on the charge is $F = QE$; the force we need to exert, by Newton’s third law, is $-QE$. So the work we do is

$$W = \int_{a}^{b} F \cdot dl = -Q \int_{a}^{b} E \cdot dl = Q \left[ V(b) - V(a) \right]$$

(independent of path). Corollary: the work required to bring charge $Q$ to point $P$ from infinity is

$$W = Q \left[ V(P) - V(\infty) \right] = QV(P)$$
Electrostatic potential energy

To obtain the **potential energy** of an assembly of charges, bring them from infinity in one by one, and calculate the work done. Consider assembling the charge distribution above: a bunch of point charges, $q_i$.

Bring in the first one: $\mathcal{W}_1 = 0$

Bring in the second one: $\mathcal{W}_2 = q_2 \left( \frac{q_1}{r_{12}} \right)$

And the third one: $\mathcal{W}_3 = q_3 \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right)$

And the fourth: $\mathcal{W}_4 = q_4 \left( \frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)$
Electrostatic potential energy (continued)

So far, for the first four charges, the total work is

\[
W = q_2 \left( \frac{q_1}{r_{12}} \right) + q_3 \left( \frac{q_1}{r_{13}} + \frac{q_2}{r_{23}} \right) + q_4 \left( \frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right)
\]

\[
= \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}
\]

Evidently, for \( N \) charges, we’d get

\[
W = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i \sum_{j=1, \, i \neq j}^{N} \frac{q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i V(P_i)
\]

(Count each pair of charges just once)
Electrostatic potential energy (continued)

If the collection of charges is finite and continuous \((N \to \infty)\), this becomes

\[
W = \frac{1}{2} \int_V dq V = \frac{1}{2} \int_V \rho V d\tau
\]

As usual, we’d also have

\[
W = \frac{1}{2} \int_S \sigma V da \quad \quad W = \frac{1}{2} \int_C \lambda V d\ell
\]

for surface- and line-charge distributions.

Eliminate density and potential from these expressions, in favor of the electric field, with

\[
\rho = \frac{1}{4\pi} \nabla \cdot E \quad \quad \nabla V = -E
\]
Electrostatic potential energy (continued)

Recall also that $\nabla \cdot (fA) = (\nabla \cdot A) f + A \cdot (\nabla f)$:

$$
\mathcal{W} = \frac{1}{8\pi} \int_{\mathcal{V}} (\nabla \cdot E)Vd\tau = \frac{1}{8\pi} \int_{\mathcal{V}} [\nabla \cdot (VE) - E \cdot \nabla V]d\tau
$$

$$
= \frac{1}{8\pi} \int_{\mathcal{V}} [\nabla \cdot (VE) + E^2]d\tau = \frac{1}{8\pi} \left[ \oint_{\mathcal{S}} VE \cdot da + \int_{\mathcal{V}} E^2 d\tau \right].
$$

Divergence theorem
Electrostatic potential energy (continued)

If we extend the integration region $\mathcal{V}$ to include all of space, and the charge distribution is finite in extent, then $E$ and $V$ approach zero at the surface $S$ (which “surrounds infinity”). Note that

$$\lim_{r \to \infty} E \propto \frac{1}{r^2}, \quad \lim_{r \to \infty} V \propto \frac{1}{r}, \quad \lim_{r \to \infty} A \propto r^2$$

$$\therefore \lim_{r \to \infty} \oint_S V E \cdot da \propto \frac{1}{r} = 0$$

Thus

$$\mathcal{W} = \frac{1}{8\pi} \int_{\text{all space}} E^2 \, d\tau = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau \quad \text{in MKS.}$$
An inconsistency?

Consider a point charge, for which the charge density is

$$\rho = q\delta^3 (r)$$

How much work is involved in assembly of this charge distribution? On the one hand,

$$W = \frac{1}{2} \int \rho V d\tau = \frac{q}{2} \int \delta^3 (r) \frac{q}{r} r^2 \sin \theta \, dr \, d\theta \, d\phi = 2\pi q^2 \int_0^\infty \delta (r) r \, dr = 0$$

but on the other,

$$W = \frac{1}{8\pi} \int E^2 d\tau = \frac{q^2}{2} \int_0^\infty \frac{1}{r^4} r^2 \, dr = -\frac{q^2}{6} \frac{1}{r^3} \bigg|_0^\infty \rightarrow \infty$$
An inconsistency? (continued)

Reason: related to the troublesome divergence of $\hat{r}/r^2$ (cf. problems 1.16, 1.38). Restore the surface integral to the expression of potential energy in terms of field:

$$W = \frac{1}{8\pi} \left[ \oint_S VE \cdot da + \int E^2\,d\tau \right] = \frac{1}{8\pi} \int\nabla \cdot (VE) + E^2\,d\tau .$$

For our point charge,

$$\nabla \cdot (VE) + E^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{q}{r^2} \right) + \frac{q^2}{r^4} = \frac{q^2}{r^2} \frac{\partial}{\partial r} \left( \frac{1}{r} \right) + \frac{q^2}{r^4}$$

$$= -\frac{q^2}{r^4} + \frac{q^2}{r^4} = 0 \quad \Rightarrow \quad W = \int\left[ \nabla \cdot (VE) + E^2 \right] d\tau = 0$$

Not, therefore, inconsistent. But use $W = 1/2 \int E^2\,d\tau$ with care, and keep that surface integral in mind.
Non-superposition of potential energy

As you know, forces, electric fields, and electric potentials obey the principle of superposition. Potential energy does not. Consider:

\[ W = \frac{1}{8\pi} \int E^2 \, d\tau = \frac{1}{8\pi} \int (E_1 + E_2 + \ldots) \cdot (E_1 + E_2 + \ldots) \, d\tau \]

\[ \neq \frac{1}{8\pi} \int (E_1^2 + E_2^2 + \ldots) \, d\tau = W_1 + W_2 + \ldots, \]

because cross terms such as

\[ \frac{1}{8\pi} \int 2E_1 \cdot E_2 \, d\tau, \quad \frac{1}{8\pi} \int 2E_1 \cdot E_3 \, d\tau, \quad \ldots \]

are not necessarily zero.