Today in Physics 217: capacitance

- Capacitance and capacitors
- Energy storage in capacitors
- Example calculations of capacitance
- Units
- Capacitors, work and stored energy
Capacitance

Consider two conductors, charged up to $Q$ and $-Q$. The potential difference between them is

$$\Delta V = -\int E \cdot dl$$

and as usual

$$E = \int \frac{\hat{r}}{r^2} \sigma \, da'$$

Suppose we double the value of $\sigma$. What happens to the other quantities?
Capacitance (continued)

Doubling $\sigma$ doubles the total charge. It also doubles the magnitude of the electric field, but not the pattern of field lines (just draw more of them). And since it doubles the field, it doubles the potential difference between the conductors.

Apparently, $dQ$ is proportional to $dV$, so $Q$ and $\Delta V$ are proportional. We call the proportionality factor the capacitance: $Q = C\Delta V$. 

![Diagram of electric field lines showing charge distribution and electric field.](image)
Energy storage in capacitors

Capacitors are important as electric circuit elements. Circuits can store energy in, and reclaim energy from, capacitors. Consider, for instance, carrying a charge $Q$ from one conductor to the other, one infinitesimal charge $dq$ at a time:

$$dW = \delta V dq = \frac{q}{C} dq$$

$$W = \frac{1}{C} \int_0^Q q dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C \Delta V^2 .$$

Usually we speak loosely about potential and potential difference in circuits, and often write $Q = CV$ or $W = CV^2 / 2$. 
Calculation of the capacitance of arrangements of conductors

Other materials besides conductors have capacitance, but arrangements of conductors lend themselves to straightforward calculation of $C$. Usually this goes as follows:

- Presume electric charge to be present; say, $Q$ if there is only one conductor, or $\pm Q$ if there are two.

- Either:
  - Calculate the electric field from the charges, and integrate it to find the potential difference $V$ between the conductors, or
  - Solve for the potential difference directly, using the integral representation or the Poisson or Laplace equation.

- Then $C = Q / V$. 

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Example calculations of capacitance

Done in the book:

- Example 2.10: parallel plates, area $A$, separation $d$:
  \[ C = \frac{A}{4\pi d} \quad \left(= \frac{\varepsilon_0 A}{d} \text{ in MKS} \right) \]

- Example 2.11: concentric spherical shells, radii $a$ and $b$:
  \[ C = \frac{ab}{b-a} \quad \left(= \frac{4\pi \varepsilon_0 ab}{b-a} \text{ in MKS} \right) \]

Both of these formulae are well worth remembering.
Example calculations of capacitance (continued)

Note that you don’t need two conductors to make a capacitor. **What is the capacitance of a conducting sphere?** Inside, \( E = 0 \); outside, \( E = \hat{r} \frac{Q}{r^2} \), so if the potential is considered to be zero at infinite distance from the sphere,

\[
V = -\int_{\infty}^{a} E \cdot dl = -Q \int_{\infty}^{a} \frac{dr}{r^2} = \frac{Q}{a}, \text{ so}
\]

\[
C = \frac{Q}{V} = a \quad (= 4\pi\varepsilon_0 a \text{ in MKS})
\]
Units of capacitance

This is a good point to bring up the question of units.

- **CGS**: the unit is the centimeter. (!) Think of the sphere.
- **MKS**: the unit is the **farad**: 
  \[ 1 \text{F} = \frac{1 \text{coul}}{1 \text{volt}} \]

Note that
\[ \varepsilon_0 = 8.85 \times 10^{-12} \text{ coul}^2 \text{ Nt}^{-1} \text{ m}^{-2} = 8.85 \times 10^{-12} \text{ F m}^{-1} \]
\[ = 8.85 \text{ pF m}^{-1} \]

One farad is a Huge capacitance. Those found around the lab are usually in the pF-μF range. On the other hand, one cm is a rather ordinary capacitance: in MKS it works out to 1.1 pF.
Work, energy storage and capacitors

Example: Griffiths problem 2.40.
Suppose the plates of a parallel-plate capacitor move closer together by an infinitesimal distance $\varepsilon$, as a result of their mutual attraction.

(a) Use what we just learned about forces on conductors to express the amount of work done by electrostatic forces, in terms of the field $E$ and the area of the plates $A$.

(b) Use $W = \frac{1}{8\pi} \int E^2 d\tau$ to express the energy lost by the field in this process.
Work, energy storage and capacitors (continued)

(a) Force on the plates:

\[ F = 2\pi \sigma^2 A \hat{n} = \frac{1}{8\pi} E^2 A \hat{n} \]

Work done by the field as it moves the plates:

\[ W_{\text{by fields}} = \int F \cdot dl = F \varepsilon = \frac{1}{8\pi} E^2 A \varepsilon \]

(b) The field is constant between the plates, so the potential energy is \( W = \frac{E^2 A d}{8\pi} \) But the distance between the plates changes to \( d - \varepsilon \):

\[ \Delta W_{\text{stored}} = \frac{1}{8\pi} E^2 A (d - (d - \varepsilon)) = \frac{1}{8\pi} E^2 A \varepsilon = W_{\text{by fields}} \]

How about that: energy is conserved.