Today in Physics 217: separation of variables IV

- Separation in cylindrical coordinates
- Example of the split cylinder (solution sketched at right)
- More on orthogonality of trig functions and Fourier’s trick

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Separation of variables in cylindrical coordinates

In Griffiths problem 3.23, on homework 6, you did the setup of the separation solution to all Laplace-equation problems in cylindrical geometry. Recall that the Laplace equation in cylindrical coordinates is

\[
\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{s^2} \frac{\partial^2 V}{\partial z^2} = 0
\]

If you know a priori that \( V \) doesn’t depend on \( z \) – infinite cylinder, boundary conditions independent of \( z \) – then the last term drops out

\[
\frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} = 0
\]

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Separation of variables in cylindrical coordinates (continued)

So you tried a solution of the form \( V(s, \phi) = S(s) \Phi(\phi) \):

\[
\frac{s}{S} \frac{d}{ds} \left( \frac{dS}{ds} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 = m^2 - m^2
\]

and solved the resulting radial and angular ordinary differential equations to obtain particular solutions:

\[
\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi \implies \Phi(\phi) = A \cos m\phi + B \sin m\phi
\]

and

\[
\frac{s}{S} \frac{d}{ds} \left( \frac{dS}{ds} \right) = m^2 \implies S(s) = C r^m + D r^{-m}
\]

with an additional radial solution for \( m = 0 \):

\[
\frac{s}{S} \frac{d}{ds} \left( \frac{dS}{ds} \right) = 0 \implies S_0(s) = C_0 \ln s + D_0
\]
Separation of variables in cylindrical coordinates (continued)

From the periodicity of the angular solution you also deduced that \( m = 0, 1, 2, 3, \ldots \)
The most general solution is a linear combination of all of these solutions, for all values of \( m \), which you wrote as

\[
V(s, \phi) = C_0 \ln s + D_0 + \sum_{m=0}^{\infty} \left( C_m s^m + D_m s^{-m} \right) \left( A_m \cos m\phi + B_m \sin m\phi \right)
\]

You applied this solution in a concrete example, Griffiths problem 3.24, in which you were able to avoid the use of Fourier’s trick. Now for one in which you can’t.

The split cylinder

Example. A long, thin-wall conducting cylindrical tube with radius \( R \) - a small section of which is shown at right - is split in half lengthwise. The two halves are insulated from one another; one is held at potential \( V_0 \) and the other is grounded. Find the potential inside the tube.

The split cylinder (continued)

The solution:

\[
V(s, \phi) = C_0 \ln s + D_0 + \sum_{m=1}^{\infty} \left( C_m s^m + D_m s^{-m} \right) \left( A_m \cos m\phi + B_m \sin m\phi \right)
\]

Boundary conditions:

i. \( V = V_0 \) at \( s = R, \phi = 0 \rightarrow \pi \)

ii. \( V = 0 \) at \( s = R, \phi = \pi \rightarrow 2\pi \)

iii. \( V \) finite at \( s = 0 \)

Apply the last one first: the term \( D_m s^{-m} \) approaches infinity at \( s = 0 \) unless \( D_m = 0 \). Similarly, \( C_0 = 0 \).
More on trig function orthogonality

Now apply the first two boundary conditions. This will be conveniently done in concert with the application of Fourier's trick to extract the rest of the coefficients inside the sum.

And this will be a good place to fill in some more details about the orthogonality of the trig functions. Last Friday we showed that

\[
\int_0^\pi \sin m\theta \sin n\theta \, d\theta = \frac{\pi}{2} \delta_{mn}
\]

Obviously we also have \( \int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta = n\delta_{mn} \).

More on trig function orthogonality (continued)

But what about cosines? It goes just like the derivation for sines, starting with two integrations by parts:

\[
\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = \int_0^{2\pi} \frac{\sin m\theta \sin n\theta}{n} \, d\theta + \int_0^{2\pi} \frac{\sin m\theta \sin n\theta}{n} \, d\theta
\]

Thus \( \int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = 0 \), or \( \int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = 0 \) \( (m \neq n) \).

For \( m = n \), \( \int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{2\pi}{2} \cos^2 \theta \)

Integrate this by parts once:

More on trig function orthogonality (continued)

\[
\int_0^{2\pi} \cos^2 \theta \, d\theta = \int_0^{2\pi} \left( \frac{1}{2} \cos 2\theta + \frac{1}{2} \sin 2\theta \right) \, d\theta = \frac{1}{2} \int_0^{2\pi} \cos 2\theta \, d\theta + \frac{1}{2} \int_0^{2\pi} \sin 2\theta \, d\theta = \frac{\pi}{2} m
\]

Thus \( \int_0^{2\pi} \cos^2 m\theta \, d\theta = \pi \)

and \( \int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta = \pi \delta_{mn} \).
More on trig function orthogonality (continued)

Then there are products of sines and cosines to consider:

\[ \int_0^{2\pi} \cos(mx) \sin(nx) \, dx = \frac{\pi}{m} \delta_{mn} \]

Recall the trig identity \( \sin(u + v) = \sin u \cos v + \cos u \sin v \):

\[ \int_0^{2\pi} \cos(mx) \sin(nx) \, dx = \frac{\pi}{m} \int_0^{2\pi} \sin(m+n)x \, dx + \frac{\pi}{m} \int_0^{2\pi} \cos mx \sin nx \, dx. \]

\[ \frac{1}{m} \int_0^{2\pi} \cos(mx) \sin(nx) \, dx = \frac{\pi}{m} \int_0^{2\pi} \cos(m+n)x \, dx = 0. \]

So, once again, this shows that

\[ \int_0^{2\pi} \cos(mx) \sin(nx) \, dx = 0 \quad (m \neq n). \]

More on trig function orthogonality (continued)

For the case \( m = n \), recall the trig identity \( 2 \cos u \sin v = \sin(2u) \):

\[ \int_0^{2\pi} \cos(mx) \sin(nx) \, dx = \frac{\pi}{m} \int_0^{2\pi} \sin(2m+n)x \, dx + \frac{\pi}{m} \int_0^{2\pi} \cos mx \sin nx \, dx. \]

So it's zero whether \( m = n \) or not.

Summary:

\[ \int_0^{2\pi} \cos(mx) \cos(nx) \, dx = \frac{\pi}{m} \delta_{mn}, \]

\[ \int_0^{2\pi} \cos(mx) \sin(nx) \, dx = 0. \]

Now back to that cylinder:

The split cylinder (continued)

What's left of the solution:

\[ V(s, \phi) = \sum_{m=1}^{\infty} \left( A_m \cos m \phi + B_m \sin m \phi \right) \]

Let's apply the first two boundary conditions, while extracting the \( A_m \):

\[ \frac{2\pi}{R} \int_0^\pi \left( A_m \cos m \phi + B_m \sin m \phi \right) \cos n \phi \, d\phi = \sum_{m=1}^{\infty} C_m^R R^m \left( A_m \cos m \phi + B_m \sin m \phi \right) \cos n \phi \, d\phi \]

\[ V_0 \int_0^\pi \cos n \phi \, d\phi = \sum_{m=1}^{\infty} C_m^R R^m \left( A_m \delta_{mn} + B_m \sin n \phi \right), \quad \text{as we just found.} \]

\[ \frac{V_0}{R} \int_0^\pi \sin n \phi \, d\phi = \pi A_m C_m^R R^m \Rightarrow A_m^R = 0 \]
The split cylinder (continued)

Now work on the $B$s by integrating everything with $\sin \phi$:

$$\frac{2\pi}{\beta} \int_0^{\beta} V(R,\phi) \sin \phi \, d\phi = \sum_{m=1}^{\infty} C_m R^m \int_0^{2\pi} (A_m \cos m\phi + B_m \sin m\phi) \sin \phi \, d\phi$$

$$V_0 \int_0^{\pi} \sin \phi \, d\phi = \sum_{m=1}^{\infty} C_m R^m \left(0 + B_m \delta_{mn}\right) = \frac{V_0}{m} = \pi B_m C_m R^m$$

$$\frac{V_0}{\pi} \int_0^{\pi} [1 - (-1)^n] = \pi B_n C_n R^n$$

$$0 \text{ if } n \text{ is even, } 2 \text{ if it's odd.}$$

The split cylinder (continued)

So replace $m$ with $2m+1$ in the sum:

$$B_n C_n = \frac{2V_0}{\pi n R^n}$$

$$V(\phi) = \sum_{m=1}^{\infty} B_m C_m \delta_{3m, n} \sin m\phi$$

$$= \frac{2V_0}{\pi} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m} \left(\frac{2m+1}{R}\right)^n \sin m\phi$$

$$= \frac{2V_0}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \left(\frac{2m+1}{R}\right)^{2m+1} \sin (2m+1)\phi$$