Today in Physics 217: electric displacement and susceptibility

- Nature of the field inside a dielectric
- Free charge and the electric displacement vector $D$
- Induced polarization and the electric susceptibility
- Confusion about susceptibility
- Calculations of $D$
- Example of the dielectric-filled parallel-plate capacitor
The field inside a dielectric

The field inside a dielectric is fantastically complicated on a microscopic level. So what is that field we calculated in the last example on Friday?

- The macroscopic, volume-averaged field, that’s what: an average taken over a size large compared to intermolecular distances in the medium. We should prove this, though (Griffiths problem 3.41 and pp. 173-175).

Consider a location \( r \) within a dielectric, and a sphere with radius \( R \) (\( \gg \) molecular sizes) centered on it. The average field within this sphere can be written as a sum of average fields produced by charges outside and inside the sphere:

\[
E_{\text{average}} (r) = E_{\text{out}} + E_{\text{in}}.
\]
The field inside a dielectric (continued)

(This is similar to problem 3.1, on homework #5, in which you showed for averages over the surface of the sphere that

\[ V_{\text{average}} = V_{\text{outside}} + V_{\text{inside}} = V_{\text{center}} + \frac{Q_{\text{enclosed}}}{R}, \]

independent of the distribution of charges within the sphere.)

First, inside. Consider a point charge at \( r^* \). The average of its field over the sphere is

\[ E_{\text{in}}^{(1)} = \frac{1}{V} \int E d\tau' \]

\[ = \frac{3}{4\pi R^3} \int \frac{q \hat{r}}{r^2} d\tau' \]
The field inside a dielectric (continued)

Now, the field from a constant charge density within the sphere is

\[
E_\rho = \int \frac{\rho \hat{\mathbf{r}}'}{r'^2} \, d\tau' = -\int \frac{\rho \hat{\mathbf{r}}}{r^2} \, d\tau' ;
\]

in other words, the point charge at a random location within the sphere has the same average field as a constant charge density

\[
\rho = -\frac{3q}{4\pi R^3} .
\]

But the field from the constant charge density is easily calculated, using Gauss’ Law (see problems 2.12, 2.18,…):
The field inside a dielectric (continued)

By superposition, then, the volume averaged field from an arbitrary distribution of charges within the sphere is

\[
E_{in}^{(1)} = E_{\rho} = \frac{4\pi \rho}{3} r'' = -\frac{4\pi}{3} \frac{3q}{4\pi R^3} r'' = -\frac{q}{R^3} = -\frac{p^{(1)}}{R^3}.
\]

By superposition, then, the volume averaged field from an arbitrary distribution of charges within the sphere is

\[
E_{in} = -\frac{p_{\text{total}}}{R^3} = -\frac{4\pi}{3} P,
\]

where \( p_{\text{total}} \) is the total dipole moment of the charges within the sphere.
The field inside a dielectric (continued)

Now for outside. Consider a point charge at $r^*$, outside the sphere this time. The average of its field over the sphere is

$$E^{(1)}_{\text{out}} = \frac{3}{4\pi R^3} \int \frac{q \hat{r}}{r'^2} \, d\tau'$$

$$= E_{\rho} = \int \frac{\rho \hat{r}'}{r'^2} \, d\tau' = -\int \frac{\rho \hat{r}'}{r'^2} \, d\tau'$$

$$= -\frac{4}{3} \pi R^3 \rho \hat{r}'' = -\frac{q^{(1)}}{r''^2} \hat{r}''$$

which is the same as the field of the charge evaluated at the center of the sphere. Thus, by superposition, $E_{\text{out}} = E_{\text{center}}$. 
The field inside a dielectric (continued)

And thus

\[ E_{\text{average}} = E_{\text{center}} - \frac{4\pi}{3} P. \]

The right-hand side is what we’ve been taking to be “the” field that we’ve calculated within dielectrics. Thus this happens to be equal to the macroscopic volume-averaged electric field.
Free charge and electric displacement

Suppose we break the total charge density in some region of space down into two parts: \( \rho = \rho_b + \rho_f \), where the bound charge \( \rho_b = -\nabla \cdot P \) is that which arises from polarization, and the free charge \( \rho_f \) is the collection of charges that has nothing to do with polarization. Then Gauss’ Law looks like

\[
\nabla \cdot E = 4\pi \rho = 4\pi \left( -\nabla \cdot P + \rho_f \right)
\]

\[
\nabla \cdot (E + 4\pi P) = 4\pi \rho_f
\]

\[
\nabla \cdot D = 4\pi \rho_f , \quad \text{where} \quad D \equiv E + 4\pi P.
\]

\[
\left[ \nabla \cdot D = \rho_f , \quad D = \varepsilon_0 E + P \text{ in MKS units.} \right]
\]

The new vector field \( D \) is called the electric displacement.
Free charge and electric displacement (continued)

- In situations in which Gauss’ Law helps, one can use this new relation to calculate $D$, and then to determine $E$ from $D$, from the free charges alone. In other words, $D$ is the same, whether or not there is polarizable material present.

- This is not as useful as it sounds, and we’ll show you why, a little later. In particular, it doesn’t really allow one to ignore the presence of polarizable media.

- The use of $D$ turns out to be most helpful in cases in which the polarization is not built in (as in our example on Friday), but instead is induced by an external applied electric field.
Induced polarization and electric susceptibility

Induced polarization is related to the local value of the externally applied electric field by

\[ P_{\text{induced}} = \chi_e E_{\text{external}} \]

where \( \chi_e \), called the electric susceptibility, is a property of the dielectric medium that is related to the atomic polarizability, molecular permanent dipole moments, etc. of its constituents.

- In general, \( \chi_e \) is a second-rank tensor. When the elements of this tensor differ, it means that a material is easier to polarize with the field in some directions than others. One can imagine how this would be true for crystals.
Induced polarization and electric susceptibility (continued)

- Special subset of physical materials: substances that polarize the same in all directions – same $|P|$ for given $|E|$, independent of the direction of $E$.

  - Such materials are called linear dielectrics, and in their case $\chi_e$ is a scalar (and $>0$), so that

    $$ D = E + 4\pi P = \left(1 + 4\pi\chi_e\right)E \equiv \varepsilon E $$

    $$ \left[D = \varepsilon_0 E + P = \varepsilon_0 \left(1 + \chi_e\right)E \equiv \varepsilon E \text{ in MKS}\right], $$

    where $\varepsilon \equiv 1 + 4\pi\chi_e \geq 1$ is the familiar dielectric constant. [Or, in MKS, $\varepsilon \equiv \left(1 + \chi_e\right)\varepsilon_0$ is the permittivity, and $K = \varepsilon / \varepsilon_0$ is the dielectric constant, identical to what is usually called $\varepsilon$ in cgs units.]
Confusion about electric susceptibility

\[ 1 + 4\pi \chi_e, \text{cgs} = \varepsilon_{\text{cgs}} = \frac{\varepsilon_{\text{MKS}}}{\varepsilon_0} = 1 + \chi_e, \text{MKS} \quad ; \text{or} \]

\[ \chi_e, \text{MKS} = 4\pi \chi_e, \text{cgs} . \]

That is, the susceptibility is dimensionless, but it’s a different dimensionless scalar in the two systems of units. Frequently, when you look susceptibility values up in tables in reference books or journal articles, you have to go to a lot of trouble to figure out which unit system is meant, as there are no units to report with the values.

Usually the cgs definition is used in the literature, even in books in which MKS dominates.

This is as bad as the cgs-MKS problems get. We will mostly avoid susceptibility, and stick to linear dielectrics.
Calculation of electric fields in situations with induced polarization

- These calculations are often simplified by calculating \( \mathbf{D} \) first and then calculating \( \mathbf{E} \):

\[
\nabla \cdot \mathbf{D} = 4\pi \rho_f \quad \Leftrightarrow \quad \oint \mathbf{D} \cdot d\mathbf{a} = 4\pi Q_f, \quad \mathbf{E} = \frac{\mathbf{D}}{\varepsilon}.
\]

- Note that induced polarization gives fields that “try” to do the same thing that free charges in conductors do: to cancel, at least partially, the applied \( \mathbf{E} \). Thus \( \mathbf{E} \) is smaller inside a dielectric immersed in an electric field, than it is outside.

\[
\begin{array}{c}
\text{Conductor} \\
\mathbf{E} = 0
\end{array}
\quad \quad
\begin{array}{c}
\text{Dielectric} \\
\mathbf{E} < \mathbf{E}_0
\end{array}
\]
The boundary conditions that apply to $D$ are not quite the same as those on $E$, because the curl of $D$ isn’t necessarily zero:

$$\nabla \times D = \nabla \times (E + 4\pi P) = 4\pi \nabla \times P$$

and the curl of $P$ doesn’t have to be zero.

Thus components of $D$ both perpendicular and parallel to surfaces are discontinuous:

$$D_{\perp,\text{above}} - D_{\perp,\text{below}} = 4\pi \sigma_f$$

$$D_{\parallel,\text{above}} - D_{\parallel,\text{below}} = 4\pi \left( P_{\parallel,\text{above}} - P_{\parallel,\text{below}} \right)$$

(Recall that the parallel component of $E$ is continuous across surfaces because the curl of $E$ is zero; see lecture notes for 27 September.)
Consider a parallel-plate capacitor with plate area $A$, separation $d$. What is its capacitance, and how does its capacitance change if one includes a dielectric slab with thickness $t$ and dielectric constant $\varepsilon$ between the plates?
Capacitor with dielectric filling (continued)

Recall that to find a capacitance, we suppose that there are charges $\pm Q$ on the conductors, calculate the electric field between the conductors, then the potential difference between the conductors; then $C = Q/V$. But this time we’ll find $E$ from $D$. Start by drawing a Gaussian cylinder that encloses some charge on one plate, and applying Gauss’ Law:

$$\oint D \cdot da = 4\pi Q_f, \text{enclosed}$$

$$D \cdot a = 4\pi \sigma_f a = 4\pi a \frac{Q}{A}$$

$$\Rightarrow D = -4\pi \frac{Q}{A} \hat{z}$$
Capacitor with dielectric filling (continued)

This value of $D$ applies everywhere between the plates, both inside and outside the dielectric slab, because the charges we assumed for the plates are the only free charges in the problem. The electric field outside and inside the slab are respectively

$$E_V = D \quad E_d = D/\varepsilon$$

Thus

$$E_V (d-t) + E_d t = V$$

$$E_V = \frac{V}{d-t + \frac{t}{\varepsilon}} = D = \frac{4\pi Q}{A}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{A}{4\pi \left( d-t + \frac{t}{\varepsilon} \right)}$$
Capacitor with dielectric filling (continued)

With no dielectric, this reduces to a result we’ve seen before:

\[ C_{\text{empty}} = \frac{A}{4\pi d} \]

If the dielectric completely fills the gap,

\[ C_{\text{full}} = \frac{\varepsilon A}{4\pi d} = \varepsilon C_{\text{empty}} \]

Since \( \varepsilon \geq 1 \), the dielectric increases the capacitance, for fixed plate size and separation.