Today in Physics 217: Ampère’s Law

- Magnetic field in a solenoid, calculated with the Biot-Savart law
- The divergence and curl of the magnetic field
- Ampère’s law
- Magnetic field in a solenoid, calculated with Ampère’s law
- Summary of electrostatics and magnetostatics so far

\[
\oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} \iint_A \mathbf{J} \cdot d\mathbf{a} = \frac{4\pi}{c} I_{\text{encl}}
\]
Another Biot-Savart law example: the solenoid

Griffiths problem 5.11: find the magnetic field at point $P$ on the axis of a tightly-wound solenoid (helical coil) consisting of $n$ turns per unit length wrapped around a cylindrical tube of radius $a$ and carrying current $I$. Express your answer in terms of $\theta_1$ and $\theta_2$ (it's easiest that way). Consider the turns to be essentially circular, and use the result of example 5.6.

What is the magnetic field on the axis of an infinite solenoid?
Reminder of the result of Example 5.6

Magnetic field a distance $z$ along the axis of a circular loop with radius $R$ and current $I$:

$$B = \hat{z} \frac{2\pi I}{c} \frac{R^2}{\left(z^2 + R^2\right)^{3/2}}$$
The solenoid (continued)

Suppose that $n$ is so large that we can consider the loops in the coil to be displaced infinitesimally; then the number of loops in a length $dz$ is $ndz$, and

$$ dB = \hat{z} \frac{2\pi I ndz}{c} \frac{a^2}{\left(z^2 + a^2\right)^{3/2}} $$
The solenoid (continued)

Take $\tan \theta = \frac{a}{z}$

so $\left(1 + \tan^2 \theta\right) d\theta = \frac{d\theta}{\cos^2 \theta} = -\frac{a}{z^2} dz = -\frac{\tan^2 \theta}{a} dz$

$\Rightarrow \ d\theta = -\frac{a}{\sin^2 \theta}$
The solenoid (continued)

\[ dB = \hat{z} \frac{2\pi In dz}{c} \left( \frac{a^2}{z^2 + a^2} \right)^{3/2} = \hat{z} \frac{2\pi In}{c} \left( -\frac{ad\theta}{\sin^2 \theta} \right) \sin^3 \theta \]

\[ = -\hat{z} \frac{2\pi In}{c} \sin \theta d\theta \; ; \]

\[ B = -\hat{z} \frac{2\pi In}{c} \int_{\theta_1}^{\theta_2} \sin \theta d\theta = \hat{z} \frac{2\pi In}{c} (\cos \theta_2 - \cos \theta_1) \]
The solenoid (continued)

For an infinite solenoid, $\theta_2 = 0$ and $\theta_1 = \pi$, so

$$B = \hat{z} \frac{2\pi I n}{c} (\cos 0 - \cos \pi) = \frac{4\pi I n}{c} \hat{z} \quad \left[= \mu_0 I n \hat{z} \text{ in MKS}\right].$$
The divergence and curl of $B$

Any vector field is uniquely specified by its divergence and curl. What are the divergence and curl of $B$?

Consider a volume $\mathcal{V}$ to contain current $I$, current density $J(r')$:

$$B(r) = \frac{1}{c} \int \frac{J(r') \times \hat{r}}{r^2} \, d\tau'$$

Denote gradient with respect to the components of $r$ and $r'$ by $\nabla$ and $\nabla'$. Now note that

$$\nabla\left(\frac{1}{r}\right) = -\nabla'\left(\frac{1}{r}\right) \quad \text{(because } r = r - r')$$

and

$$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}.$$
The divergence and curl of $B$ (continued)

With these,

$$B(r) = -\frac{1}{c} \int \frac{\hat{\mathbf{r}}}{\mathbf{r}^2} \times J(r') \, d\tau' = \frac{1}{c} \int \nabla \left( \frac{1}{\mathbf{r}} \right) \times J(r') \, d\tau'$$

$$= \frac{1}{c} \nabla \times \int \frac{J(r')}{\mathbf{r}} \, d\tau' \quad \text{(remember, } J \neq f(r)\text{)}.$$

This is a useful form for $B$, which we will use a lot next lecture too (the integral turns out to be the magnetic vector potential, $A$). Take its divergence:

$$\nabla \cdot B(r) = \frac{1}{c} \nabla \cdot \left( \nabla \times \int \frac{J(r')}{\mathbf{r}} \, d\tau' \right) = 0.$$

The divergence of any curl is zero, remember.
The divergence and curl of $B$ (continued)

Integrate this last expression over any volume: 

$$\int \nabla \cdot B(r) \, d\tau = \oint B \cdot da = 0 \ .$$

Compare these to the expressions for $E$ in electrostatics, and we see that magnetostatics involves no counterpart of charge: there's no "magnetic charge."

Now for the curl:

$$\nabla \times B(r) = \frac{1}{c} \nabla \times \nabla \times \int \frac{J(r')}{r} \, d\tau' \ .$$

Use Product Rule #10:

$$\nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A \ :$$
The divergence and curl of $B$ (continued)

\[
\nabla \times B(r) = \frac{1}{c} \nabla \left( \nabla \cdot \int_{\mathcal{V}} J(r') \frac{d\tau'}{r} \right) - \frac{1}{c} \nabla^2 \int_{\mathcal{V}} \frac{J(r')}{r} d\tau'
\]

\[
= \frac{1}{c} \nabla \left( \int_{\mathcal{V}} \nabla \cdot J(r') d\tau' \right) - \frac{1}{c} \int_{\mathcal{V}} J(r') \nabla^2 \left( \frac{1}{r} \right) d\tau'.
\]

Now use your old friend Product Rule #5,

\[
\nabla \cdot (fA) = f (\nabla \cdot A) + A \cdot (\nabla f)
\]

to write

\[
\nabla \cdot \left( \frac{J(r')}{r} \right) d\tau' = \left( \frac{1}{r} \right) \nabla \cdot J(r') + J(r') \cdot \nabla \left( \frac{1}{r} \right) = J(r') \cdot \nabla \left( \frac{1}{r} \right)
\]

= 0 (J independent of $r$)
The divergence and curl of $B$ (continued)

Also, 
\[ \nabla^2 \left( \frac{1}{\rho} \right) = \nabla \cdot \nabla \left( \frac{1}{\rho} \right) = \nabla \cdot \left( \frac{\hat{\rho}}{\rho^2} \right) = 4\pi \delta^3 (\rho) , \]

so 
\[ \nabla \times B(r) = \frac{1}{c} \nabla \int J(r') \cdot \nabla \left( \frac{1}{\rho} \right) d\tau' + \frac{4\pi}{c} \int J(r') \delta^3 (r - r') d\tau' \]

\[ = -\frac{1}{c} \nabla \int J(r') \cdot \nabla' \left( \frac{1}{\rho} \right) d\tau' + \frac{4\pi}{c} J(r) . \]

Use Product Rule #5 again, on the first term:

\[ J(r') \cdot \nabla' \left( \frac{1}{\rho} \right) = \nabla' \cdot \left( \frac{J(r')}{\rho} \right) - \frac{1}{\rho} \nabla' \cdot J(r') = \nabla' \cdot \left( \frac{J(r')}{{\rho}} \right) \]

=0 in magnetostatics
The divergence and curl of $B$ (continued)

So,

$$\nabla \times B(r) = -\frac{1}{c} \nabla \left( \int V' \left( \frac{J(r')}{\mathbf{r}} \right) d\tau' \right) + \frac{4\pi}{c} J(r)$$

$$= -\frac{1}{c} \nabla \left( \oint_S \frac{J(r')}{\mathbf{r}} \cdot d\mathbf{a}' \right) + \frac{4\pi}{c} J(r) .$$

But by definition $J = 0$ on the surface, so the integral vanishes:

$$\nabla \times B(r) = \frac{4\pi}{c} J(r) . \quad \textbf{Ampère's Law}$$

This can be put into integral form by choosing an area that some current flows through:
The divergence and curl of $B$ (continued)

\[
\int_A (\nabla \times B) \cdot da = \frac{4\pi}{c} \int_A J \cdot da
\]

\[
\oint_C B \cdot dl = \frac{4\pi}{c} I_{\text{enclosed}}.
\]

Ampère's law is to magneto-statics what Gauss' law is to electrostatics, except that one uses an Ampèrean loop to enclose current, instead of a Gaussian surface to enclose charge. The same tricks we learned with Gauss' law and superposition have analogues in magnetostatics.
Example: field in an infinite solenoid

The symmetry of the coil dictates that the field must be along $z$, and must be a lot stronger inside than out, so if the number of turns per unit length is $n$, and the current is $I$,

\[ \oint_C \mathbf{B} \cdot d\ell = \frac{4\pi}{c} \int_A \mathbf{J} \cdot d\mathbf{a} = \frac{4\pi}{c} I_{\text{enclosed}} \]

\[ B\Delta z = \frac{4\pi}{c} I n\Delta z \quad \Rightarrow \quad B = \frac{4\pi nI}{c} \hat{z} \quad [= \mu_0 nI \text{ in MKS}]. \]

Same as before!
Maxwell’s equations for electrostatics and magnetostatics

Note the similarities and differences:

\[ \nabla \cdot \mathbf{E} = 4\pi \rho \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = 0 \quad \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \]

\[ \oint \mathbf{E} \cdot d\mathbf{a} = 4\pi Q_{\text{encl}} \quad \oint \mathbf{B} \cdot d\mathbf{a} = 0 \]

\[ \oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \oint \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} I_{\text{encl}} \]
Maxwell’s equations for electrostatics and magnetostatics

Note the similarities and differences (MKS):

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad \nabla \cdot B = 0$$

$$\nabla \times E = 0 \quad \nabla \times B = \mu_0 J$$

$$\oint E \cdot da = \frac{1}{\varepsilon_0} Q_{\text{encl}} \quad \oint B \cdot da = 0$$

$$\oint E \cdot d\ell = 0 \quad \oint B \cdot d\ell = \mu_0 I_{\text{encl}}$$