Today in Physics 217: magnetoquasistatics

- Magnetoquasistatics
- Example magnetoquasistatic calculations with Faraday’s Law
- Faraday-induced electric fields and the magnetic vector potential
- When does a uniformly charged spherical shell have nonzero $E$ inside?

*Flux tubes on the Sun, seen by NASA’s TRACE satellite.*
Magnetoquasistatics

With Faraday’s Law, we can pose lots of new problems in which one generates time-variable magnetic fields, and calculates electric fields. We have derived formulas for $B$ generated by lots of current distributions and we can use...

- But can we? All those calculations assume steady currents, and give steady magnetic fields.
- Thus, to use them straight away with time-varying currents gives magnetic fields with the same time dependence everywhere in space.
- This can’t be right! If I look far enough away from the currents I should see a magnetic field that depends on what the current was a little while ago, because the electromagnetic information of what the current is, can propagate to that point no faster than the speed of light.
Magnetoquasistatics (continued)

- So any calculation we do in this manner gives only an approximate solution, good only if the current doesn’t fluctuates very fast or if we’re not calculating the field very far away. If $\tau$ is the time over which the current changes and $r$ is the distance away from the currents at which we want to calculate the field, then we need to stay within

$$r \ll c\tau.$$  
**Magnetoquasistatic approximation**

- You will learn how to do it right next semester in PHY 218, using **retarded fields** and **retarded potentials**. But for now we’ll assume we’re to be working within the quasistatic approximation.
Magnetoquasistatics (continued)

For instance, consider **Example 7.9**: “An infinitely long straight wire carries a current $I(t)$. Determine the induced electric field, a distance $s$ from the wire.” The result is

$$E = \frac{2}{c} \frac{dI}{dt} \ln \left( \frac{s}{s_0} \right) \hat{z} \xrightarrow{s \to \infty} \infty.$$

The fact that it blows up as $s \to \infty$ is an indication of the violation of the magnetoquasistatic approximation at distances too great for the field to be in sync with the current. See Example 10.2 for the sorts of things that are necessary to get the right answer. (See it next semester, though.)
Electric field from a variable solenoidal current

Griffiths problem 7.15

An infinite solenoid with radius $a$ and $n$ turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\phi}$ direction. Find the electric field at a distance $s$ from the axis, inside and out.

The magnetic field is zero outside and

$$B = \frac{4\pi}{c} n I \hat{z}$$

inside, so the flux through a circular loop with radius $s$ is

$$\Phi_B = \begin{cases} 
\frac{4\pi}{c} n I \pi s^2 & (s < a) \\
\frac{4\pi}{c} n I \pi a^2 & (s > a)
\end{cases}$$
Electric field from a variable solenoidal current (continued)

And
\[ \oint E \cdot d\ell = -\frac{1}{c} \frac{d\Phi_B}{dt} \Rightarrow E2\pi s = \begin{cases} -\frac{4\pi}{c^2} s^2 n \frac{dI}{dt} & (s < a) \\ -\frac{4\pi}{c^2} a^2 n \frac{dI}{dt} & (s > a) \end{cases} \]

so
\[ E = \begin{cases} -\frac{2\pi}{c^2} s n \frac{dI}{dt} \hat{\phi} & (s < a) \\ -\frac{2\pi}{c^2} a^2 n \frac{dI}{dt} \hat{\phi} & (s > a) \end{cases} \]

For MKS units, change one of the factors of $1/c$ to $\mu_0/4\pi$, and the other to unity.
Breaking a wire

Griffiths problem 7.18

A square conducting loop, with side $a$ and resistance $R$, lies a distance $a$ from an infinite straight wire that carries current $I$. Suppose that we break the wire, so that $I$ drops abruptly to zero. What total charge passes a given point in the loop during the time this current flows, and in what direction does the induced current in the square loop flow?

Field from the straight infinite current:

$$B = \frac{2I}{cs} \hat{\phi}.$$
Breaking a wire (continued)

So the flux of $B$ through the square loop is

$$
\Phi_B = \frac{2Ia}{c} \int_a^{2a} \frac{ds}{s} = \frac{2Ia}{c} \ln s \bigg|_a^{2a} = \frac{2Ia}{c} \ln 2 \quad , \text{ and}
$$

$$
\mathcal{E} = I_{\text{loop}} R = \frac{dQ}{dt} R = - \frac{1}{c} \frac{d\Phi_B}{dt} = - \frac{2a \ln 2}{c^2} \frac{dI}{dt} .
$$

Note that $Q$ depends on the current in the loop, not $I$. And it’s $Q$ we want:

$$
Q = \int_0^Q dQ' = - \frac{2a \ln 2}{c^2 R} \int_0^l dI' = \frac{2Ia \ln 2}{c^2 R} \left[ = \frac{\mu_0 Ia \ln 2}{2\pi R} \right] \text{ in MKS}.
$$

It doesn’t matter how fast the current shuts off…
Breaking a wire (continued)

As for the direction of the induced current:
Before the wire is cut, the field points out of the page.
Shutting off the current decreases this field. The current in the loop will flow so as to oppose this change (Lenz’s Law.) If the induced current flows **counterclockwise,** it will produce a magnetic field in the same direction as the (decreasing) field from the wire, which is what it takes to oppose the change.
Faraday-induced electric fields and the magnetic vector potential

In magnetostatics,

\[ \nabla \cdot B = 0 \quad , \quad \nabla \times B = \frac{4\pi}{c} J \quad , \quad B = \frac{1}{c} \int \frac{J(r') \times \hat{n}}{\tau^2} d\tau'. \]

For **induced** electric fields and \( \rho = 0 \), we can substitute

\[ B \leftrightarrow E \quad , \quad J \leftrightarrow -\frac{1}{4\pi} \frac{\partial B}{\partial t} , \]

and get

\[ \nabla \cdot E = 0 \quad , \quad \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \quad , \quad E = -\frac{1}{4\pi c} \frac{\partial}{\partial t} \int \frac{B(r',t) \times \hat{n}}{\tau^2} d\tau'. \]
Faraday-induced electric fields and the magnetic vector potential (continued)

Furthermore, \[ \nabla \cdot A = 0 \quad , \quad \nabla \times A = B \quad , \]

so \( A \) depends on \( B \) in the same way that \( B \) depends on \( \frac{4\pi}{c} J \).

Thus

\[
A = \frac{1}{4\pi} \int \frac{B(r', t) \times \hat{n}}{r'^2} \, d\tau' , \quad \text{and}
\]

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t} .
\]

Check: \[ \nabla \times E = -\frac{1}{c} \frac{\partial}{\partial t} \nabla \times A = -\frac{1}{c} \frac{\partial B}{\partial t} \quad \text{(Faraday's Law again).} \]

Leave off the factor of \( 1/c \) for MKS.
When does a uniformly charged spherical shell have nonzero $E$ inside?

A spherical shell with radius $R$ carries a uniform charge density $\sigma$. It spins about the $z$ axis at angular velocity $\omega(t)$ that changes with time. Find the electric field inside and outside the sphere. (Griffiths problem 7.47)

The week before last (Example 5.11, lecture, 11 November 2002), we worked out the magnetic vector potential from a spinning, uniformly charged spherical shell, and got

$$A(r) = \begin{cases} \frac{4\pi R\sigma\omega}{3c}r \sin \theta \hat{\phi} & (r < R), \\ \frac{4\pi R^4\sigma\omega}{3c} \frac{\sin \theta}{r^3} \hat{\phi} & (r > R). \end{cases}$$

$A$ is time dependent if $\omega$ is.
When does a uniformly charged spherical shell have nonzero $E$ inside? (continued)

So there are two contributions to the electric field: electrostatic, and induced. The static part is, of course,

$$E_{\text{static}}(r) = \begin{cases} 0 & (r < R), \\ \frac{4\pi R^2 \sigma}{r^2} \hat{r} & (r > R). \end{cases}$$

Using the $E$-$A$ relation we just derived, the induced part is

$$E_{\text{ind}}(r, t) = -\frac{1}{c} \frac{\partial A}{\partial t} = \begin{cases} -\frac{4\pi R \sigma}{3c^2} \frac{d\omega}{dt} r \sin \theta \hat{\phi} & (r < R), \\ -\frac{4\pi R^4 \sigma}{3c^2} \frac{d\omega}{dt} \frac{\sin \theta}{r^3} \hat{\phi} & (r > R). \end{cases}$$
When does a uniformly charged spherical shell have nonzero $E$ inside? (continued)

So the total electric field is

$$E(r, t) = \begin{cases} 
-\frac{4\pi R\sigma}{3c^2} \frac{d\omega}{dt} r \sin\theta \hat{\varphi} & (r < R) \\
\frac{4\pi R^2\sigma}{r^2} \hat{r} - \frac{4\pi R^4\sigma}{3c^2} \frac{d\omega}{dt} \frac{r^3}{r^3} \hat{\varphi} & (r > R)
\end{cases}$$

, or

$$E(r, t) = \begin{cases} 
-\frac{\mu_0 R\sigma}{3} \frac{d\omega}{dt} r \sin\theta \hat{\varphi} & (r < R) \\
\frac{R^2\sigma}{\varepsilon_0 r^2} \hat{r} - \frac{\mu_0 R^4\sigma}{3} \frac{d\omega}{dt} \frac{r^3}{r^3} \hat{\varphi} & (r > R)
\end{cases}$$

in MKS.

It’s nonzero inside, if the angular velocity changes with time.