Today in Physics 217: AC circuits

- Finish resistive circuit problem from last time
- The LRC circuit
- Energy in LRC, and the meaning of Quality

The third simplest AC circuit: LRC

Suppose that, in this simple circuit, the charge on the capacitor is \( q_0 \), and the current \( I = 0 \), at \( t = 0 \). What is the charge on the capacitor at later times?

Set up first:
- Let upper plate have positive charge initially, and define \( I \) flowing into this plate, so that \( I = dq/dt \).
- With this choice of direction for \( I \), the polarities for the potential differences across \( L \) and \( R \) are as shown.
- The magnitudes of the potential differences are \( V_C = q/C \), \( V_R = IR \), and \( V_L = LdI/dt \) (back EMF).

LRC (continued)

Now apply Kirchhoff’s second rule:

\[
\frac{dq}{dt} + IR + L \frac{dI}{dt} = 0 \quad \text{or} \quad \frac{dq}{dt} + \frac{R}{L} \frac{dq}{dt} + \frac{d^2q}{dt^2} = 0
\]

and define two useful new quantities:

\[
\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(natural frequency)}, \quad Q = \frac{\omega_0 L}{R} \quad \text{(quality)}
\]

so that

\[
\frac{d^2q}{dt^2} + \frac{\omega_0}{Q} \frac{dq}{dt} + \omega_0^2 q = 0
\]
Many of you will recognize this as the equation of motion of a damped harmonic oscillator, and will be able to recite the solution without my help. Too bad, here it is anyway. Use an exponential trial solution:

\[ q = Ae^{\omega t} , \]
\[ \frac{dq}{dt} = \rho A e^{\omega t} = \rho q , \]
\[ \frac{d^2q}{dt^2} = \rho^2 A e^{\omega t} = \rho^2 q . \]

Thus

\[ \rho^2 - \frac{\omega_0^2}{Q^2} \rho + \omega_0^2 = 0 . \]

This is just a quadratic equation, and its solutions are

\[ \rho = \frac{\omega_0}{2Q} \pm \sqrt{\frac{\omega_0^2}{4Q^2} - \frac{\omega_0}{Q}} \sqrt{1 - \left( \frac{1}{2Q} \right)^2} . \]

The general solution to the differential equation is thus

\[ q(t) = A \exp \left( \frac{\omega_0 t}{2Q} + i \omega_0 \sqrt{\frac{1}{2Q} - \frac{1}{4Q^2}} \right) \]
\[ + B \exp \left( - \frac{\omega_0 t}{2Q} - i \omega_0 \sqrt{\frac{1}{2Q} - \frac{1}{4Q^2}} \right) . \]

Now write \( D = \sqrt{1 - (1/2Q)^2} \), and apply the initial conditions, \( q(0) = q_0 \) and \( \frac{dq}{dt}(0) = 0 \):

\[ q(0) = A + B = q_0 \]
\[ \frac{dq}{dt}(0) = \left( - \frac{\omega_0}{2Q} + i \omega_0 D \right) A + \left( - \frac{\omega_0}{2Q} - i \omega_0 D \right) B = 0 , \] so
\[ \begin{cases} - \frac{\omega_0}{2Q} + i \omega_0 D \ A + \left( - \frac{\omega_0}{2Q} - i \omega_0 D \right) (q_0 - A) = 0 , \\ (2i \omega_0 D) A = \frac{\omega_0}{2Q} + i \omega_0 D q_0 , \end{cases} \] and...
LRC (continued)

\[ A \left( 1 - \frac{i}{2QD} \right) \frac{q_0}{2}, \quad B = q_0 - A \left( 1 + \frac{i}{2QD} \right) \frac{q_0}{2}. \]

So the general solution becomes

\[ q(t) = A + B = q_0 \exp \left( -\frac{\omega D}{2Q} t \right) \exp \left( \frac{i\omega D}{2Q} t \right). \]

Rearrange the complex exponentials and you get sines and cosines:

\[ q(t) = q_0 \exp \left( -\frac{\omega D}{2Q} t \right) \left[ 1 + \frac{i}{2QD} \left( \frac{1}{2} e^{\omega D t} - \frac{1}{2i} e^{-\omega D t} \right) \right]. \]

It is useful to consider the limit of high quality, \( Q \gg 1 \):

\[ D = \sqrt{1 - \left( \frac{1}{2Q} \right)^2} \approx 1 \quad \text{to first order in} \quad \frac{1}{Q}, \]

so \( q(t) = q_0 \exp \left( -\frac{\omega D}{2Q} t \right) \left[ \cos \omega D t + \frac{1}{2Q} \sin \omega D t \right]. \)

Now use \( \cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \), and note that since \( Q \gg 1 \), \( \sin(1/2Q) \approx 1/2Q \) and \( \cos(1/2Q) \approx 1 \) to first order:

\[ q(t) = q_0 \exp \left( -\frac{\omega D}{2Q} t \right) \cos \left( \omega D t - \frac{1}{2Q} \right). \]
This is a damped oscillation: less strongly damped the larger \( Q \) (or the smaller \( R \)) is. Its period is \( T = 2\pi/\omega_0 \).

Energy in LRC, and the significance of quality

At the extrema of \( q \), the current is zero, so the total energy of the circuit is stored in the capacitor, as \( W_{\text{max}} = q^2/2C \).

- Between extrema, the peak charge decreases by a factor of \( e^{-\pi \omega T/2Q} = e^{-\pi Q} \), so the energy decreases by the square of this factor, or \( e^{-2\pi Q} \).
- Energy is conserved, so the energy dissipated during one period of the oscillation is
  \[
  \Delta W = W_{\text{max}} \left(1 - e^{-2\pi Q}\right).
  \]

Energy in LRC, and the significance of quality (continued)

If \( Q \gg 1 \), then

\[
\frac{W_{\text{max}}}{\Delta W} = \frac{1}{1 - e^{-2\pi Q}} \approx \frac{1}{\left(1 - \left(\frac{2\pi}{Q}\right)^2\right)^{1/2}} = \frac{Q}{2}\pi.
\]

Thus an interpretation of \( Q \) emerges:

\[
Q = \frac{2\pi}{\Delta W} \quad \frac{W_{\text{max}}}{\Delta W} = \frac{2\pi}{\Delta W} 
\]

maximum stored energy

energy dissipated per period

\[
= \frac{W_{\text{max}}}{\Delta W} = \text{energy dissipated per radian}.
\]