Today in Physics 217: resonant AC circuits

- Finish the “calculation” from last time
- The driven series LRC circuit
- Phasors
- Complex impedance
- Resonance
Driven \textit{LRC} circuit

Consider now an \textit{LRC} circuit, in which a sinusoidally varying EMF is imposed on the series combination:

\[ V_L = V_I = V_R = V \]

As usual, the question is: what is the current, as a function of time?
Driven LRC circuit (continued)

Kirchhoff’s Rule #2 says:

\[ \mathcal{E}_0 \cos \omega t = L \frac{dI}{dt} + IR + \frac{q}{C}. \]

Let’s just go ahead and try a solution similar to what we obtained for the un-driven circuit last Wednesday:

\[ I = I_0 \cos(\omega t + \phi_0). \]

For this current, the charge on the capacitor would be

\[ q = \frac{I_0}{\omega} \sin(\omega t + \phi_0) \quad \Rightarrow \quad \frac{dq}{dt} = I_0 \cos(\omega t + \phi_0) = I, \]

so

\[ \mathcal{E}_0 \cos \omega t = -\omega LI_0 \sin(\omega t + \phi_0) + R I_0 \cos(\omega t + \phi_0) + \frac{I_0}{\omega C} \sin(\omega t + \phi_0). \]
Driven LRC circuit (continued)

Recall $\cos(\theta \pm \pi/2) = \mp \sin \theta$:

$$\mathcal{E}_0 \cos \omega t = \omega LI_0 \cos \left(\omega t + \phi_0 + \frac{\pi}{2}\right) + RI_0 \cos(\omega t + \phi_0)$$

$$+ \frac{I_0}{\omega C} \cos \left(\omega t + \phi_0 - \frac{\pi}{2}\right)$$

, or

$$\text{Re}\left(\mathcal{E}_0 e^{i\omega t}\right) = \text{Re}\left(\omega LI_0 e^{i\omega t} e^{i\phi_0} e^{i\pi/2} + RI_0 e^{i\omega t} e^{i\phi_0}ight)$$

$$+ \frac{I_0}{\omega C} e^{i\omega t} e^{i\phi_0} e^{-i\pi/2}$$

, or even

$$\text{Re}(\mathcal{E}_0) = \text{Re}\left(\omega LI_0 e^{i\phi_0} e^{i\pi/2} + RI_0 e^{i\phi_0} + \frac{I_0}{\omega C} e^{i\phi_0} e^{-i\pi/2}\right)$$.
Driven LRC circuit (continued)

To find $I_0$ from here, we merely need to multiply this equation by its complex conjugate. But there are two fruitful ways to think about what this means, so even though the next steps are simple we will use them to illustrate both concepts:

- Phasors
- Complex impedance
Phasors

Plotting the complex numbers (for which our solution is only the real part!) graphically makes obvious an analogy between them and vectors, and makes the meaning of our next step easier. A complex number $\tilde{x} = \rho e^{i\phi}$ can be represented in polar coordinates in the complex plane with radius given by its magnitude and azimuth given by its argument. Their addition is rather like addition of the vectors that result.
Phasors (continued)

We don’t call them vectors in the complex plane, though (although they rotate like vectors in that space); we call them phasors. In phasor-speak, the loop equation for our circuit is

\[ \tilde{\mathcal{E}}_0 = \tilde{V}_L + \tilde{V}_R + \tilde{V}_C \]

In terms of the magnitudes of the phasors,

\[ \mathcal{E}_0^2 = \tilde{V}_L^2 + \tilde{V}_R^2 + \tilde{V}_C^2 = R^2 I_0^2 + \left( \frac{1}{\omega C} - \omega L \right)^2 I_0^2 \]

\[ \tan \phi_0 = \left( \frac{1}{\omega C} - \omega L \right) / R \]
Complex impedance

Back to that previous expression:

\[ \text{Re}(\mathcal{E}_0) = \text{Re} \left( \omega LI_0 e^{i\phi_0} e^{i\pi/2} + RI_0 e^{i\phi_0} + \frac{I_0}{\omega C} e^{i\phi_0} e^{-i\pi/2} \right) \]

\[ = \text{Re} \left( i\omega LI_0 e^{i\phi_0} + RI_0 e^{i\phi_0} + \frac{I_0}{i\omega C} e^{i\phi_0} \right) \]

\[ = \text{Re} \left( ZI_0 e^{i\phi_0} \right) , \text{ where} \]

\[ Z = Z_R + Z_L + Z_C = R + i\omega L + \frac{1}{i\omega C} . \]

Z is called the complex impedance. Three terms add in this series circuit, just as the resistance of three series resistors would.
Complex impedance

That this comes from our previous result, illustrates the fact that we can use complex impedance in a complex form of Ohm’s law, just as a (real) resistance:

\[ Z = R + i\omega L + \frac{1}{i\omega C} \]

\[ V = \mathcal{E}_0 e^{i\omega t} \]

\[ I = \frac{V}{Z} = \frac{\mathcal{E}_0 e^{i\omega t}}{R + i\left(\omega L - \frac{1}{\omega C}\right)} \]

This of course involves a complex current, so we’d still have to remember to take the real part at the end.
Driven LRC circuit (continued)

Either way, we have a complex current given by

\[
I = \frac{R - i \left( \omega L - \frac{1}{\omega C} \right)}{R^2 + \omega^2 L^2 \left( 1 - \frac{1}{\omega^2 LC} \right)^2} \mathcal{E}_0 e^{i \omega t} = \frac{\mathcal{E}_0 e^{i \omega t}}{R} \left[ 1 - i \frac{\omega L}{R} \left( 1 - \frac{1}{\omega^2 LC} \right) \right]
\]

\[
= \frac{\mathcal{E}_0 e^{i \omega t}}{R} \frac{1 - i \frac{\omega Q}{\omega_0} \left( 1 - \frac{\omega_0^2}{\omega^2} \right)}{1 + \left( \frac{\omega Q}{\omega_0} \right)^2 \left( 1 - \frac{\omega_0^2}{\omega^2} \right)^2} = I_{\text{in phase}} + I_{\text{quadrature}}.
\]

The in-phase (real) component is maximum at \( \omega = \omega_0 \).
Resonance

The amplitude of the in-phase current is a very sharply-peaked function if \( Q \gg 1 \). This phenomenon – in which a system is driven at its natural frequency – is called **resonance**.

\[
\frac{I_0 R}{\mathcal{E}_0}
\]
Resonance (continued)

It turns out to be interesting to calculate the full width at half maximum (FWHM) in the case $Q \gg 1$. If we note that the peak is very sharp, so that $\omega$ is close to the natural frequency if the current is very different from zero, then

$$1 - \frac{\omega_0^2}{\omega^2} = \frac{\omega^2 - \omega_0^2}{\omega^2} = \frac{(\omega + \omega_0)(\omega - \omega_0)}{\omega^2} \approx \frac{2\omega_0(\omega - \omega_0)}{\omega_0^2} = 2 \frac{\omega - \omega_0}{\omega_0}$$

and the amplitude of the in-phase current is

$$I_{IP,0} \approx \frac{\mathcal{E}_0}{R} \frac{1}{1 + 4Q^2(\omega - \omega_0)^2/\omega_0^2} = \frac{\mathcal{E}_0}{2R}$$

for half maximum;

$$1 + 4Q^2(\omega - \omega_0)^2/\omega_0^2 = 2$$
Resonance (continued)

This is just a quadratic, easily solved:

\[ 4Q^2 \left( \omega - \omega_0 \right)^2 / \omega_0^2 = 1 \]

\[ \omega^2 - 2\omega\omega_0 + \omega_0^2 \left( 1 - \frac{1}{4Q^2} \right) = 0 \]

\[ \omega = \frac{2\omega_0 \pm \sqrt{4\omega_0^2 - 4\omega_0^2 \left( 1 - \frac{1}{4Q^2} \right)}}{2} = \omega_0 \pm \frac{\omega_0}{2Q} \]

\[ FWHM = \omega_0 + \frac{\omega_0}{2Q} - \left( \omega_0 - \frac{\omega_0}{2Q} \right) = \frac{\omega_0}{Q} \]
Resonance (continued)

Thus another significance of the quality $Q$: the width (at least the FWHM) of a resonance is proportional to $1/Q$. 

![Graph showing the relationship between $I_0 R / \mathcal{E}_0$ and $\omega / \omega_0$]