Today in Physics 218: gauge transformations

More updates as we move from quasistatics to dynamics:

- #3: potentials
- For better use of potentials: gauge transformations
- The Coulomb and Lorentz gauges
- #4: force, energy, and momentum in electrodynamics

The spectre of the Brocken. Photo by Galen Rowell.
Update #3: potentials

In electrodynamics the divergence of $B$ is still zero, so according to the Helmholtz theorem and its corollaries (#2, in this case), we can still define a magnetic vector potential as

$$B = \nabla \times A.$$

However, the curl of $E$ isn’t zero; in fact it hasn’t been since we started magnetoquasistatics. What does this imply for the electric potential? Note that Faraday’s law can be put in a suggestive form:

$$\nabla \times E = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times A), \text{ or }$$

$$\nabla \times \left( E + \frac{1}{c} \frac{\partial A}{\partial t} \right) = 0.$$
Potentials (continued)

Thus Corollary #1 to the Helmholtz theorem allows us to define a scalar potential for that last bracketed term:

\[ E + \frac{1}{c} \frac{\partial A}{\partial t} = -\nabla V \quad \Rightarrow \quad E = -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t} \]

so we can still use the scalar electric potential in electrodynamics, but now both the scalar and the vector potential must be used to determine \( E \).
"Reference points" for potentials

Our usual reference point for the scalar potential in electrostatics is $V \to 0$ at $r \to \infty$. For the vector potential in magnetostatics we imposed the condition $\nabla \cdot A = 0$.

- These reference points arise from exploitation of the built-in ambiguities in the static potentials: one can add any gradient to $A$ and any constant to $V$, and still get the same fields.
- So we decided to add whatever was necessary to make the second-order differential equations in $A$ and $V$ look like Poisson’s equation (i.e. easy to solve).

In electrodynamics these choices no longer produce that last result:
“Reference points” for potentials (continued)

For instance, Gauss’s law gives us
\[ \nabla \cdot E = 4\pi \rho \quad \Rightarrow \quad \nabla \cdot \left( -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t} \right) = 4\pi \rho \]

\[ \Rightarrow \nabla^2 V + \frac{1}{c} \frac{\partial}{\partial t} \nabla \cdot A = -4\pi \rho , \]

which with \( \nabla \cdot A = 0 \) still leaves us with a Poisson equation, but Ampère’s law gives
\[ \nabla \times \left( \nabla \times A \right) = \frac{4\pi}{c} J - \frac{1}{c} \frac{\partial}{\partial t} \nabla V - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \]

\[ \nabla \left( \nabla \cdot A \right) - \nabla^2 A = \quad \text{(P.R. #11)} \]

or
\[ \left( \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \right) - \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial V}{\partial t} \right) = -\frac{4\pi}{c} J \quad . \]
“Reference points” for potentials (continued)

This latter equation does not of course reduce to a Poisson equation with any of the reference conditions we have imposed. Thus we must look harder to use the built-in ambiguity of the potentials to make the differential equations simpler. The general way to do this, which we will cover next time, is called a gauge transformation.
Gauge transformations

In electro- and magentostatics, we showed that we could always choose our conventional reference points,

\[ V \to 0 \text{ as } r \to \infty \quad \nabla \cdot A = 0 \]

without placing any peculiar constraints on \( E \) or \( B \). Now we have two, more complicated equations to simplify, and a more general approach is more fruitful.

Consider performing a transformation on \( A \) and \( V \): add a vector to \( A \) and a scalar to \( V \), giving new potential functions:

\[ A' = A + \alpha \quad V' = V + \beta \]
Gauge transformations (continued)

Now, we can’t add just any old thing to the potentials; we need for the fields arising from the new potentials to be the same as those from the old:

\[ B' = B \]
\[ \nabla \times A' = \nabla \times A + \nabla \times \alpha \]
\[ \nabla \times \alpha = 0 \quad , \text{or} \quad \alpha = \nabla \lambda' \]

\[ E' = E \]
\[ \nabla V' = \nabla V + \nabla \beta \]
\[ -E' - \frac{1}{c} \frac{\partial A'}{\partial t} = -E - \frac{1}{c} \frac{\partial A}{\partial t} + \nabla \beta \]

\[ \nabla \beta = \frac{1}{c} \frac{\partial}{\partial t} \left( A - A' \right) = -\frac{1}{c} \frac{\partial \alpha}{\partial t} \]

where \( \lambda' \) is a scalar function of \( r \) and \( t \).
Gauge transformations (continued)

Combine those last two results:

\[ \nabla \beta = -\frac{1}{c} \frac{\partial}{\partial t} \nabla \lambda' \]

\[ \nabla \left( \beta + \frac{1}{c} \frac{\partial \lambda'}{\partial t} \right) = 0 \]

and integrate the second one over volume, applying the fundamental theorem of calculus:

\[ \beta + \frac{1}{c} \frac{\partial \lambda'}{\partial t} = f(t) \]

The integration “constant” \( f \) is not a function of position.
Gauge transformations (continued)

We can combine the integration “constant” \( f \) with \( \lambda' \) by defining

\[
\lambda = \lambda' - c \int_{0}^{t} f(t') \, dt
\]

Then,

\[
\beta = -\frac{1}{c} \frac{\partial \lambda'}{\partial t} + f(t) = -\frac{1}{c} \left( \frac{\partial \lambda}{\partial t} + cf(t) \right) + f(t) = -\frac{1}{c} \frac{\partial \lambda}{\partial t}
\]

\[
\alpha = \nabla \lambda' = \nabla \lambda
\]

Thus for any scalar function \( \lambda = \lambda(r,t) \), the transformation

\[
V' = V - \frac{1}{c} \frac{\partial \lambda}{\partial t} \quad A' = A + \nabla \lambda
\]

makes new potentials but leaves the fields \( E \) and \( B \) unchanged.
Gauge transformations (continued)

This sort of operation on potentials is called a gauge transformation, and a particular choice of \( \lambda \) is called a gauge condition.

- Clever choices of \( \lambda \) can simplify one or the other of the second-order differential equations for the potentials.
- The solution of these simpler equations for the transformed potentials gives the same fields as the solutions to the untransformed, complicated equations.

For instance, to simplify

\[
\nabla^2 V + \frac{1}{c} \frac{\partial}{\partial t} \nabla \cdot A = -4\pi \rho ,
\]

we could pick \( \lambda \) such that

\[
\nabla \cdot A = 0
\]

Coulomb gauge
Gauge transformations (continued)

which, as we showed in PHY 217 (see http://www.pas.rochester.edu/~dmw/phy217/Lectures/Lect_28b.pdf) we can always do; that is, there always exists a function $\lambda$ that we can add to $A$ to give an $A'$ with zero divergence. With this gauge condition,

$$\nabla^2 V' = -4\pi \rho,$$

just like in electrostatics (hence the name).

- Coulomb gauge only does a lot of good in magnetoquasistatics, because otherwise the time derivative of $A$ gets big enough that you have to remember that

$$E = -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t}.$$
Lorentz gauge

It’s hard to compute $A$ in Coulomb gauge. On the other hand, we could choose $\lambda$ such that

$$\nabla \cdot A + \frac{1}{c} \frac{\partial V}{\partial t} = 0$$

for which the second-order PDEs we saw several pages back become

$$\nabla^2 V + \frac{1}{c} \frac{\partial}{\partial t} \nabla \cdot A = -4\pi\rho$$

$$\Rightarrow \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -4\pi\rho$$
Lorentz gauge (continued)

and

\[
\left( \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \right) - \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial V}{\partial t} \right) = -\frac{4\pi}{c} J
\]

\[
\Rightarrow \quad \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} J
\]

- Not utterly simple, but at least \(V\) and \(A\) are separately determined, and the four equations are very similar to one another.

- In fact, the four second-order PDEs here are (inhomogeneous) wave equations, the solution of which will concern us for the bulk of this semester.
Can one always use the Lorentz gauge? I think so, because:

\[ \nabla \cdot A' + \frac{1}{c} \frac{\partial V'}{\partial t} = 0 \]

\[ \nabla \cdot (A + \nabla \lambda) + \frac{1}{c} \frac{\partial}{\partial t} \left( V - \frac{1}{c} \frac{\partial \lambda}{\partial t} \right) = 0 \]

\[ \nabla^2 \lambda - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} = -\left( \nabla \cdot A + \frac{1}{c} \frac{\partial V}{\partial t} \right) = g(r, t) \]

That is, the Lorentz gauge condition \( \lambda \) always obeys an inhomogenous wave equation, just as do the potentials in Lorentz gauge. In MTH 281 you proved the existence of solutions to such equations; we’ll demonstrate such solutions this semester.
Update #4: force, energy, and momentum in electrodynamics

The Lorentz force law,

\[ F = q \left( E + \frac{1}{c} v \times B \right) \]

hasn’t changed since we first learned it, but can be used to illuminate the relation of the potentials to mechanics. To wit:

\[ F = q \left( -\nabla V - \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} v \times \nabla \times A \right) \]

According to product rule #4, written for \( v \) and \( A \),

\[ \nabla (v \cdot A) = v \times (\nabla \times A) + A \times (\nabla \times v) + (v \cdot \nabla) A + (A \cdot \nabla) v \]

\[ = v \times (\nabla \times A) + (v \cdot \nabla) A \quad \text{because } v \text{ doesn’t depend explicitly on } r \]
Force, energy, and momentum in electrodynamics (continued)

so

\[
F = -q \left( \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} (\mathbf{v} \cdot \nabla) A + \nabla \left( V - \frac{1}{c} \mathbf{v} \cdot A \right) \right)
\]

It will be useful to introduce total time derivatives:

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial y} \frac{dy}{dt} + \frac{\partial A}{\partial z} \frac{dz}{dt}
\]

\[
= \frac{\partial A}{\partial t} + \left( v_x \frac{\partial A}{\partial x} + v_y \frac{\partial A}{\partial y} + v_z \frac{\partial A}{\partial z} \right) A = \frac{\partial A}{\partial t} + (\mathbf{v} \cdot \nabla) A
\]

so

\[
F = -q \left( \frac{1}{c} \frac{dA}{dt} + \nabla \left( V - \frac{1}{c} \mathbf{v} \cdot A \right) \right)
\]
Force, energy, and momentum in electrodynamics (continued)

But \( F = \frac{dp}{dt} \), where \( p \) is the momentum of the point charge \( q \), so

\[
\frac{dp}{dt} = -q \left( \frac{1}{c} \frac{dA}{dt} + \nabla \left( V - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right) \right)
\]

\[
\frac{d}{dt} \left( p + \frac{q}{c} \mathbf{A} \right) = -\nabla \left( q \left[ V - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right] \right)
\]

This has the form \( F = \frac{d}{dt} p_{\text{canonical}} = -\nabla U \), where the canonical momentum is

\[ p_{\text{canonical}} = p + \frac{q}{c} \mathbf{A} \]

Since \( \mathbf{v} = \frac{dr}{dt} \), \( r \) is the conjugate canonical coordinate, and the related potential energy is

\[ U = q \left( V - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right) \]
Force, energy, and momentum in electrodynamics (continued)

In quantum mechanics one normally uses the Hamiltonian formulation of dynamics; this last result represents the easiest way to incorporate electrodynamics into quantum mechanics. From correspondence to classical mechanics, the Hamiltonian

\[ H = \dot{x}_{\text{canonical}} \cdot p_{\text{canonical}} - L \]

gives rise to the quantum-mechanical equations of motion,

\[ H\psi = E\psi \]

as well as the classical ones,

\[ \frac{\partial H}{\partial p_{\text{canonical}}} = \dot{x}_{\text{canonical}} \quad , \quad \frac{\partial H}{\partial x_{\text{canonical}}} = \dot{p}_{\text{canonical}} \]
The classical Lagrangian for a charge $q$ in an electromagnetic field is therefore

$$L = \frac{1}{2} m \dot{x}_{\text{canonical}}^2 - U = \frac{1}{2} m \dot{v}^2 - qV + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}$$

so the classical Hamiltonian is

$$H = \dot{x}_{\text{canonical}} \cdot \mathbf{p}_{\text{canonical}} - L = m \dot{v}^2 + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} - \frac{1}{2} m \dot{v}^2 + qV - \frac{q}{c} \mathbf{v} \cdot \mathbf{A}$$

$$= \frac{1}{2} m \dot{v}^2 + qV = \frac{p^2}{2m} + qV = \frac{1}{2m} \left| \mathbf{p}_{\text{canonical}} - \frac{q}{c} \mathbf{A} \right|^2 + qV \ .$$

In quantum mechanics, $\mathbf{p}_{\text{canonical}} \rightarrow -i\hbar \nabla$.