Today in Physics 218: stratified linear media I

- Interference in layers of linear media
- Transmission and reflection in stratified linear media, viewed as a boundary-value problem
- Matrix formulation of the fields at the interfaces
Interference in layers of linear media

As a preamble to the general question of transmission and reflection by stratified media, we will ask a simpler one: what is the condition for completely constructive interference in a single layer of linear material?

- Consider two plane-parallel, partially reflecting surfaces separated by a linear medium with refractive index $n = \sqrt{\mu \varepsilon}$ and thickness $d$ (next slide).
- It doesn’t matter what the index of refraction outside the reflectors is, but we will assume here that it is unity (vacuum) on both sides.
- If the transmitted or reflected rays are focussed then the waves interfere. By calculating the path-length differences, we can find out how they interfere.
Interference in layers of linear media (continued)

\[ \theta_t = \sin^{-1}\left( \frac{\sin \theta_i}{n} \right) \]

\[ n = \sqrt{\mu \varepsilon} \]
Interference in layers of linear media (continued)

- The path length difference between any two successive transmitted waves is the same. For the first set, that’s the length between \( AB \) and \( ACD \):
  \[
  AB = 2d \tan \theta_t \sin \theta_i = 2dn \frac{\sin^2 \theta_t}{\cos \theta_t},
  \]
  \[
  n \sin \theta_t = \sin \theta_i \]
  \[
  AD = 2d \tan \theta_t \implies ACD = \frac{2d}{\cos \theta_t}.
  \]

- The wavelength is \( \lambda \) in vacuum and \( \lambda/n \) in the medium between the reflectors, so
  \[
  \delta(AB) = 2\pi \frac{AB}{\lambda} = \frac{4\pi dn}{\lambda \cos \theta_t} \sin^2 \theta_t,
  \]
  \[
  \delta(ACD) = 2\pi \frac{nACD}{\lambda} = \frac{4\pi dn}{\lambda \cos \theta_t}.\]
Interference in layers of linear media (continued)

- If the phase difference is an integer multiple of $2\pi$, then the interference between the two wavefronts corresponding to these paths is completely constructive:

$$\Delta \delta = \delta (ACD) - \delta (AB) = \frac{4\pi dn}{\lambda \cos \theta_t} \left(1 - \sin^2 \theta_t \right) = \frac{4\pi dn \cos \theta_t}{\lambda} ,$$

$$= 2\pi m \quad (m = 0, 1, 2, \ldots).$$

- Thus there are maxima in the spectrum of the transmission of the dielectric slab, at wavelengths given by

$$\lambda_m = \frac{2dn \cos \theta_t}{m} \quad (m = 0, 1, 2, \ldots).$$

This, BTW, is the principle of the Fabry-Perot interferometer.
Transmission and reflection in stratified linear media, viewed as a boundary-value problem

Now we will set up the general solution to the problem of the transmission and reflection by a plane parallel layer, and find thereby a method for dealing with as many layers as we want.

Consider light propagating in one medium, incident obliquely on a layer of a second medium, and emerging into a third (next slide). What are the amplitudes of the transmitted and reflected waves?

- As before, this can be broken into two parts, one with light polarized perpendicular to the plane of incidence (TE), and one with $E$ parallel to the plane of incidence (TM). We’ll do TE first, and fill out the boundary conditions at the surfaces.
Transmission and reflection in stratified linear media as a boundary-value problem (continued)

\( \varepsilon_0, \mu_0 \) 1 \( \varepsilon_1, \mu_1 \) 2 \( \varepsilon_2, \mu_2 \)

\( \tilde{E}_I \) \( \theta_I \) \( \tilde{B}_I \) \( k_{I1} \)

\( \tilde{B}_T \) \( \tilde{E}_T \) \( \theta_{T1} \) \( k_{T1} \)

\( \tilde{B}_R \) \( \tilde{E}_R \) \( \theta_I \) \( k_{R1} \)

\( \tilde{B}_{T2} \) \( \tilde{B}_{T2} \) \( \theta_{T2} \) \( k_{T2} \)

\( d \)

TE waves
Transmission and reflection in stratified linear media as a boundary-value problem (continued)

- The electric fields look generically like this:
  \[
  \tilde{E} = \tilde{E}_0 e^{i(nk \cdot r - \omega t)} \quad \text{for waves propagating toward } +z, \\
  \tilde{E} = \tilde{E}_0 e^{i(-nk \cdot r - \omega t)} \quad \text{the other way.}
  \]

And of course \( \tilde{B} = \sqrt{\mu \varepsilon} \hat{k} \times \tilde{E} \).

- At surface 1, the boundary conditions on \( E_\parallel \) and \( H_\parallel \) are
  \[
  \tilde{E}_\parallel,1 = \tilde{E}_{0I} + \tilde{E}_{0R1} = \tilde{E}_{0T1} + \tilde{E}_{0R2}, \\
  \tilde{H}_\parallel,1 = \frac{1}{\mu_0} \left( \tilde{B}_{0I} \cos \theta_I - \tilde{B}_{0R1} \cos \theta_I \right) \\
  = \frac{1}{\mu_1} \left( \tilde{B}_{0T1} \cos \theta_{T1} - \tilde{B}_{0R2} \cos \theta_{T1} \right).
  \]
Transmission and reflection in stratified linear media as a boundary-value problem (continued)

or

\[
\tilde{E}_{0I} + \tilde{E}_{0R1} = \tilde{E}_{0T1} + \tilde{E}_{0R2},
\]

\[
\sqrt{\frac{\varepsilon_0}{\mu_0}} \cos \theta_I (\tilde{E}_{0I} - \tilde{E}_{0R1}) = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} (\tilde{E}_{0T1} - \tilde{E}_{0R2}).
\]

Next the wave traverses the layer filled with medium #1, as follows:

![Diagram showing wave traversal through a layered medium](image_url)
Transmission and reflection in stratified linear media as a boundary-value problem (continued)

As a wave crosses the slab it travels a distance

\[ PQ = d / \cos \theta_{T_1} \]

Compared to the undisplaced wave that would have resulted if the slab were not there, it undergoes a phase change of

\[
\delta_1 = k_1 \ell_1 + k_2 \ell_2 = \frac{2\pi n_1}{\lambda} PQ - \frac{2\pi n_2}{\lambda} RS
\]

\[
= \frac{2\pi n_1 d}{\lambda \cos \theta_{T_1}} - \frac{2\pi n_2}{\lambda} d \tan \theta_{T_1} \sin \theta_{T_2}
\]

\[
= \frac{2\pi n_1 d}{\lambda \cos \theta_{T_1}} \left(1 - \sin^2 \theta_{T_1}\right) = \frac{2\pi n_1 d}{\lambda} \cos \theta_{T_1}
\]

(half that of the two reflections in slide 5)
Transmission and reflection in stratified linear media as a boundary-value problem (continued)

Thus the $E_\parallel$ and $H_\parallel$ boundary conditions at surface 2 are

$$\tilde{E}_{\parallel, 2} = \tilde{E}_{0T1} e^{i\delta_1} + \tilde{E}_{0R2} e^{-i\delta_1} = \tilde{E}_{0T2} e^{i\delta_1},$$

$$\tilde{H}_{\parallel, 2} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \left( \tilde{E}_{0T1} e^{i\delta_1} - \tilde{E}_{0R2} e^{-i\delta_1} \right) = \sqrt{\frac{\varepsilon_2}{\mu_2}} \cos \theta_{T2} \tilde{E}_{0T2} e^{i\delta_1}.$$

At this point we have four equations that we can solve for the four unknown amplitudes, $\tilde{E}_{0R1}$, $\tilde{E}_{0R2}$, $\tilde{E}_{0T1}$, and $\tilde{E}_{0T2}$, for the TE case. You can proceed directly in this manner, to solve a couple of the problems in this week’s homework (e.g. Crawford 5.21, Griffiths !9.34). But it would be incredibly tedious to treat more than one layer like this. Fortunately there’s a better way…
Matrix formulation of the fields at the interfaces

The clever way to solve these problems starts by rearranging the boundary conditions to obtain relations between the fields at the two interfaces. \( \tilde{E}_{0T1} \) and \( \tilde{E}_{0R2} \) appear in both sets of boundary conditions, so solve the latest result for these two amplitudes:

\[
\sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \tilde{E}_{\parallel,2} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \left( \tilde{E}_{0T1} e^{i\delta_1} + \tilde{E}_{0R2} e^{-i\delta_1} \right)
\]

\[
\tilde{H}_{\parallel,2} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \left( \tilde{E}_{0T1} e^{i\delta_1} - \tilde{E}_{0R2} e^{-i\delta_1} \right)
\]

\[
\sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \tilde{E}_{\parallel,2} + \tilde{H}_{\parallel,2} = 2 \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \tilde{E}_{0T1} e^{i\delta_1} , \text{ or}
\]
Matrix formulation of the fields at the interfaces (continued)

\[ \tilde{E}_{0T1} = \frac{1}{2} e^{-i\delta_1} \left( \tilde{E}_{||,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{||,2}}{\cos \theta_{T1}} \right) . \]

- Put this back in the surface-2 boundary conditions, and solve for \( \tilde{E}_{0R2} \):

\[ \tilde{E}_{||,2} = \frac{1}{2} e^{-i\delta_1} \left( \tilde{E}_{||,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{||,2}}{\cos \theta_{T1}} \right) e^{i\delta_1} + \tilde{E}_{0R2} e^{-i\delta_1} , \text{ or} \]

\[ \tilde{E}_{0R2} = \frac{1}{2} e^{i\delta_1} \left( \tilde{E}_{||,2} - \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{\tilde{H}_{||,2}}{\cos \theta_{T1}} \right) . \]

- Now put both of these into the surface-1 boundary conditions:
Matrix formulation of the fields at the interfaces
(continued)

\[
\tilde{E}_{\|,1} = \frac{1}{2} e^{-i\delta_1} \left( \tilde{E}_{\|,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \tilde{H}_{\|,2} \right) + \frac{1}{2} e^{i\delta_1} \left( \tilde{E}_{\|,2} - \sqrt{\frac{\mu_1}{\varepsilon_1}} \tilde{H}_{\|,2} \right)
\]

\[
= \tilde{E}_{\|,2} \cos \delta_1 - \tilde{H}_{\|,2} \sqrt{\frac{\mu_1}{\varepsilon_1}} \frac{i \sin \delta_1}{\cos \theta_{T1}}
\]

\[
\tilde{H}_{\|,1} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} \left[ \frac{1}{2} e^{-i\delta_1} \left( \tilde{E}_{\|,2} + \sqrt{\frac{\mu_1}{\varepsilon_1}} \tilde{H}_{\|,2} \right) - \frac{1}{2} e^{i\delta_1} \left( \tilde{E}_{\|,2} - \sqrt{\frac{\mu_1}{\varepsilon_1}} \tilde{H}_{\|,2} \right) \right]
\]

\[
= -\tilde{E}_{\|,2} \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} i \sin \delta_1 + \tilde{H}_{\|,2} \cos \delta_1
\]
Matrix formulation of the fields at the interfaces (continued)

Now define

\[ Y_{1,TE} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \cos \theta_{T1} = \frac{4\pi}{c} \frac{1}{Z_1} \cos \theta_{T1} \]

and the results look suggestive of matrix arithmetic:

\[ \begin{align*}
\tilde{E}_{\parallel,1} &= \tilde{E}_{\parallel,2} \cos \delta_1 - \tilde{H}_{\parallel,2} \frac{i \sin \delta_1}{Y_{1,TE}} \\
\tilde{H}_{\parallel,1} &= -\tilde{E}_{\parallel,2} Y_{1,TE} i \sin \delta_1 + \tilde{H}_{\parallel,2} \cos \delta_1
\end{align*} \]

or

\[
\begin{bmatrix}
\tilde{E}_{\parallel,1} \\
\tilde{H}_{\parallel,1}
\end{bmatrix} =
\begin{bmatrix}
\cos \delta_1 & -i \sin \delta_1 / Y_{1,TE} \\
-\frac{i Y_{1,TE} \sin \delta_1}{Y_{1,TE} \sin \delta_1} & \cos \delta_1
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_{\parallel,2} \\
\tilde{H}_{\parallel,2}
\end{bmatrix}
\equiv M_1
\begin{bmatrix}
\tilde{E}_{\parallel,2} \\
\tilde{H}_{\parallel,2}
\end{bmatrix}.
\]

\( M_1 \) is called the characteristic matrix of layer 1.
Matrix formulation of the fields at the interfaces (continued)

☐ We could repeat this procedure for TM waves (see following slide), but it’s so similar to what we just did that we’ll just skip to the result:

\[
\tilde{E}_{\parallel,1} = -\tilde{H}_{\parallel,2} \sqrt{\frac{\mu_1}{\varepsilon_1}} \cos \theta_{T1} i \sin \delta_1 + \tilde{E}_{\parallel,2} \cos \delta_1 ,
\]

\[
\tilde{H}_{\parallel,1} = \tilde{H}_{\parallel,2} \cos \delta_1 - \tilde{E}_{\parallel,2} \sqrt{\frac{\varepsilon_1}{\mu_1}} i \sin \delta_1 .
\]

☐ Thus if we define

\[
\gamma_{1,TM} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \frac{1}{\cos \theta_{T1}} ,
\]

we get the same matrix equation as before, which we will write as:
Matrix formulation of the fields at the interfaces (continued)

\[ \varepsilon_0, \mu_0 \quad \varepsilon_1, \mu_1 \quad \varepsilon_2, \mu_2 \]

\[ \begin{align*}
\mathbf{B}_I & \quad \mathbf{E}_I \quad k_{I1} \\
\mathbf{E}_{R1} & \quad \mathbf{H}_{R1} \quad k_{R1} \\
\mathbf{B}_{T1} & \quad \mathbf{E}_{T1} \quad k_{T1} \\
\mathbf{E}_{R2} & \quad \mathbf{H}_{R2} \quad k_{R2} \\
\mathbf{B}_{T2} & \quad \mathbf{E}_{T2} \quad k_{T2} \\
d &
\end{align*} \]

TM waves
Matrix formulation of the fields at the interfaces (continued)

\[
\begin{bmatrix}
\tilde{E}_{\|,1} \\
\tilde{H}_{\|,1}
\end{bmatrix}
= \begin{bmatrix}
\cos \delta_1 & -i \sin \delta_1 / \gamma_1 \\
-i \gamma_1 \sin \delta_1 & \cos \delta_1
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_{\|,2} \\
\tilde{H}_{\|,2}
\end{bmatrix}
\equiv M_1 \begin{bmatrix}
\tilde{E}_{\|,2} \\
\tilde{H}_{\|,2}
\end{bmatrix}.
\]

If there were yet a third surface to the right, the parallel components of the fields there could therefore be determined from

\[
\begin{bmatrix}
\tilde{E}_{\|,2} \\
\tilde{H}_{\|,2}
\end{bmatrix}
= M_2 \begin{bmatrix}
\tilde{E}_{\|,3} \\
\tilde{H}_{\|,3}
\end{bmatrix},
\]

which can be combined with our first result to yield

\[
\begin{bmatrix}
\tilde{E}_{\|,1} \\
\tilde{H}_{\|,1}
\end{bmatrix}
= M_1 M_2 \begin{bmatrix}
\tilde{E}_{\|,3} \\
\tilde{H}_{\|,3}
\end{bmatrix}.
\]
Matrix formulation of the fields at the interfaces (continued)

And so on. Evidently, for a stack of \( p \) layers, the parallel components of the fields at the first and \( p+1 \)th surface are related by

\[
\begin{bmatrix}
\tilde{E}_{\parallel,1} \\
\tilde{H}_{\parallel,1}
\end{bmatrix} = M_1 M_2 \cdots M_p \begin{bmatrix}
\tilde{E}_{\parallel,p+1} \\
\tilde{H}_{\parallel,p+1}
\end{bmatrix},
\]

and the whole stack can be said to have a characteristic matrix \( M \) given by

\[
M = M_1 M_2 \cdots M_p = \begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}.
\]