Today in Physics 217: magnetic multipoles

- Multipole expansion of the magnetic vector potential
- Magnetic dipoles
- Magnetic field from a magnetic dipole
- Torque on a magnetic dipole in uniform $B$
- Force and energy and magnetic dipoles
Multipole expansion of the magnetic vector potential

Consider an arbitrary loop that carries a current $I$. Its vector potential at point $r$ is

$$A(r) = \frac{I}{c} \oint \frac{d\ell}{r}.$$  

Just as we did for $V$, we can expand $1/r$ in a power series and use the series as an approximation scheme:

$$\frac{1}{r} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r'}{r} \right)^n P_n(\cos\theta)$$

(see lecture notes for 21 October 2002 for derivation).
Multipole expansion of the magnetic vector potential (continued)

Put this series into the expression for $A$:

$$A(r) = \frac{I}{c} \left[ \frac{1}{r} \oint d\ell + \frac{1}{r^2} \oint r' \cos \theta \, d\ell \right. $$

$$+ \frac{1}{r^3} \oint r'^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) d\ell$$

$$+ \frac{1}{r^4} \oint r'^3 \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) d\ell$$

$$+ \ldots \right]$$

Of special note in this expression:
Multipole expansion of the magnetic vector potential (continued)

- The monopole term is zero, since

\[ \oint d\ell = 0. \]

This isn’t surprising, since “no magnetic monopoles” is built into the Biot-Savart law, from which we obtained \( A \).

- For points far away from the loop compared to its size, we obtain a good approximation for \( A \) by using just the first (or first two) nonvanishing terms. (For points closer by, one would need more terms for the same accuracy.)

- This is, of course, the same useful behaviour we saw in the multipole expansion of \( V \).
Magnetic dipoles

\[ A_{\text{dipole}}(r) = \frac{I}{cr^2} \oint r' \cos \theta \, d\ell = \frac{I}{cr^2} \oint (\hat{r} \cdot r') \, d\ell \]

Note that

\[ d \left[ (\hat{r} \cdot r')r' \right] = (\hat{r} \cdot dr')r' + (\hat{r} \cdot r') \, dr' \]

\[ \oint d \left[ (\hat{r} \cdot r')r' \right] = \oint (\hat{r} \cdot dr')r' + \oint (\hat{r} \cdot r') \, dr' \]

but \[ \oint d \left[ (\hat{r} \cdot r')r' \right] = 0 \], so

\[ \oint (\hat{r} \cdot dr')r' = -\oint (\hat{r} \cdot r') \, dr' \] .

Also,

\[ \hat{r} \times \oint r' \times dr' = \oint r'(\hat{r} \cdot dr') - \oint (\hat{r} \cdot r') \, dr' = -2\oint (\hat{r} \cdot r') \, dr' \] .
Magnetic dipoles (continued)

Thus
\[ \oint (\hat{r} \cdot r') dr' = \oint (\hat{r} \cdot r') d\ell = -\frac{1}{2} \hat{r} \times \oint r' \times d\ell \]

so
\[ A_{\text{dipole}} = -\frac{1}{2} \frac{I}{cr^2} \hat{r} \times \oint r' \times d\ell \]
\[ = -\frac{\hat{r} \times m}{r^2} = \frac{m \times \hat{r}}{r^2} \]

where
\[ m \equiv \frac{I}{2c} \oint r' \times d\ell . \]

(Compare to \( V_{\text{dipole}} = \frac{p \cdot \hat{r}}{r^2} \), \( p = \int \rho r' d\tau' \).)
Magnetic dipoles (continued)

To see what this means in terms of geometry of a current loop and its dipole moment, consider the triangle formed by \( r' \) and \( d\ell \):

\[
\left| \frac{1}{2} r' \times d\ell \right| = \frac{1}{2} r'd\ell \sin\alpha = \frac{1}{2} r'd\ell \sin(\pi - \alpha)
\]

= area of triangle.

So

\[
\frac{1}{2} r' \times d\ell = da', \quad \text{and}
\]

\[
m = \frac{I}{c} \oint da' = \frac{I}{c} a \quad \text{for plane loops.}
\]