Let $f : V \to R$ be a linear function; i.e., an element of $V'$. Then $f(u)$ can be thought of as a linear function from $V' \to R$ if we hold $u$ fixed and vary $f$. Thus each element $u \in V$ defines a linear function $V' \to R$. In other words we have a natural inclusion $V \subset (V')'$.

If $V$ is finite dimensional, we can expand any vector in some basis $e_1 \cdots e_n$. Any linear function is determined uniquely by its values $f_i = f(e_{(i)})$ on this basis:

$$u = \sum_i u^i e_{(i)}, \quad f(u) = \sum_i u^i f_i. \quad (1)$$

It follows that any linear function on $V'$ is of the form $\sum_i u^i f_i$ for some sequence of numbers, and hence that $(V')' = V$.

Define the function

$$f(t) = g(u + tv, u + tv) = g(u, u) + 2tg(u, v) + t^2g(v, v). \quad (2)$$

For a positive inner product, this is always positive. By applying calculus of one variable we find its minimum value to be at $t_0 = -\frac{g(u, v)}{g(v, v)}$, taking the value

$$f(t_0) = g(u, u) - \frac{g(u, v)^2}{g(v, v)} \quad (3)$$

which must be positive. Multiplying by $g(v, v)$ gives the required answer.

3.1

$$dx = dr \cos \theta - \sin \theta rd\theta, \quad dy = dr \sin \theta + r \cos \theta d\theta. \quad (4)$$

Applying trig identities, we get

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \quad (5)$$

Recall that the Laplacian in arbitrary co-ordinates is

$$\Delta \psi = \frac{1}{\sqrt{g}} \partial_i [\sqrt{g}g^{ij} \partial_j \psi] \quad (6)$$

We have $\sqrt{g} = r$, $g^{rr} = 1$, $g^{r\theta} = 0$, $g^{\theta\theta} = r^{-2}$, to get

$$\Delta \psi = \frac{1}{r^2} \partial^2_{\theta} \psi + \frac{1}{r} \partial_r [r \partial_r \psi] \quad (7)$$
If the Schrödinger equation \(-\Delta \psi + V\psi = E\psi\) is to be separable, the solution must have the form \(\psi(r,\theta) = R(r)\Theta(\theta)\); i.e.,

\[
\frac{1}{rR(r)} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} R \right] + \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} + V(r,\theta) = E
\]  \hspace{1cm} (8)

Thus the potential must have the form

\[
V(r,\theta) = A(r) + \frac{B(\theta)}{r^2}.
\]  \hspace{1cm} (9)

The Laplace equation is the special case \(V = 0\).
The three cases:

$S^1$: Choose the usual polar angle $\theta_1$ for the neighborhood $S^1 - \{\theta_1 = 0\}$ and another angular coordinate $\theta_2$ which is zero at any point other than that for which $\theta_1$ is zero for the neighborhood $S^1 - \{\theta_2 = 0\}$.

$S^1 \times S^1$: Decompose the torus into two circles with one circle in the plane of the torus representing the position along the inner radius and the second perpendicular circle the position around the circumference. Then attach a copy of the atlas constructed above to each circle with inner circle coordinates $\{\theta_1, \phi_1\}$ and outer coordinates $\{\theta_2, \phi_2\}$. The four charts comprising the atlas of the torus are then given by $(\theta_1, \phi_1)$, $(\theta_2, \phi_1)$, $(\theta_1, \phi_2)$, $(\theta_2, \phi_2)$.

$S^2$: Choose the usual polar angle $\theta_1$ and its corresponding azimuthal angle $\phi_1$ for the neighborhood $S^2 - \{\phi_1 = 0, \theta_1 = 0, \theta_1 = \pi\}$ and another chart $\theta_2$ and $\phi_2$ with the curve represented by $\phi_2 = 0$ of the second chart rotated about the point represented by $\theta_1 = \frac{\pi}{2}, \phi_1 = 0$ by $\frac{\pi}{2}$ anticlockwise and then about $\theta_1 = 0$ by $\pi$ for the neighborhood $S^2 - \{\phi_2 = 0, \theta_2 = 0, \theta_2 = \pi\}$.

4.1 Geometrically, wrap the real line around the circle. This constructs a multi-valued map $\phi : S^1 \mapsto \mathbb{R}^1$ which then induces another multi-valued map from the coordinates on $S^1$ to the coordinates on $\mathbb{R}^1$, for instance $\theta \mapsto \{x + 2\pi n\}$, $n \in \mathbb{I}$. It also maps each function on $S^1$ to a function on $\mathbb{R}^1$: for fixed $\theta$ and $x$,

\[ f : \theta \mapsto f(\theta) \mapsto g : \{x + 2\pi n\} \mapsto f(\theta) \]

\[ f \in C(S^1), \ g \in C(\mathbb{R}^1) \]

Therefore, each such function $g$ has period $2\pi$. Next, consider the single-valued map $\psi : \mathbb{R}^1 \mapsto S^1$. This map in turn induces a single-valued map from the coordinates on $\mathbb{R}^1$ to the coordinates on $S^1$ and a map from each function with period $2\pi$ on $\mathbb{R}^1$ to a function on $S^1$.

\[ g : \{x + 2\pi n\} \mapsto g(x) \mapsto f : \theta \mapsto g(x) \]

It is now clear that $\psi$ can be identified with $\phi^{-1}$. Therefore, $\phi$ is one-to-one and onto with respect to $2\pi$-periodic functions. This establishes a one-to-one correspondence between functions on $S^1$ and $2\pi$-periodic functions on $\mathbb{R}^1$. 

3
4.2 The vector field \( \frac{d}{d\theta} \) does not vanish on \( S^1 \) except in the degenerate limit that the radius \( R \) of the circle goes to zero. This may be avoided by using a field such as \( v(\theta) \equiv \frac{1}{2\pi} \frac{d}{d\theta} \) which, when written in terms of Cartesian coordinates that are valid in this limit, clearly blows up at the origin and cannot otherwise be zero:

\[
v(\theta(x, y)) = \frac{-y}{x^2 + y^2} \partial_x + \frac{x}{x^2 + y^2} \partial_y
\]

5 Such a function is represented by:

\[
\begin{cases}
1 & \text{if } |x| \leq \epsilon \\
1 - \exp \left[ -\frac{1}{(|x|^2 - \epsilon^2)} \right] \exp \left[ \frac{1}{|x|^2 - \frac{1}{(\epsilon^2 - \sigma^2)}} \right] & \text{if } \epsilon < |x| < \sigma \\
0 & \text{if } |x| \geq \sigma
\end{cases}
\]

where \( \sigma = \epsilon + \epsilon^2 \). There cannot exist such an analytic function; for instance, a Taylor series expansion around any point farther from the origin than \( \epsilon + \epsilon^2 \) would give the function to be the zero function everywhere.

6.1 This operation is clearly antisymmetric. Direct substitution and expansion then establishes the Jacobi identity:

\[
[[u, v] \cdot w] + [[v, w] \cdot u] + [[w, u] \cdot v] = [uv - vu, w] + [vw - wv, u] + [wu - uw, v]
\]

\[
= uvu - wuv - vwu + wuv + vwu - wuv - wvu + wuv + wvu - wuv + wuv = 0
\]

6.2 The vector cross product makes \( \mathbb{R}^3 \) into a Lie algebra. That it satisfies the Jacobi identity can be seen by expanding with the vector triple product rule and using the commutativity of the inner product:

\[
(X \times Y) \times Z = Y(X \cdot Z) - X(Y \cdot Z)
\]

\[
Y(X \cdot Z) - X(Y \cdot Z) + Z(Y \cdot X) - Y(Z \cdot X) + X(Z \cdot Y) - Z(X \cdot Y) = 0
\]

7 Addition of derivations is defined by the sum’s action on a real-valued function as follows:

\[
(X + Y)(f) \equiv X(f) + Y(f)
\]

Since \( X(f) \) and \( Y(f) \) are real-valued functions in \( C(M) \), their addition is commutative and associative. It then follows that addition of derivations is also commutative and associative:

\[
(X + Y)(f) = X(f) + Y(f) = Y(f) + X(f) = (Y + X)(f)
\]

\[
((X + Y) + Z)(f) = (X(f) + Y(f)) + Z(f) = X(f) + (Y(f) + Z(f)) = (X + (Y + Z))(f)
\]
Also, there exists a null derivation \(= 0\) that acts as the additive identity. Next, by the definition of linear, the action of a derivation on a sum of functions is distributive across addition of functions. Finally, addition of derivations was defined by the statement that it is distributive across addition of derivations. Therefore, the space of derivations over a field is a real vector space. As any function defined on the manifold can be part of a derivation, this vector space has infinite dimension.

Alternately, note that derivations are a subset of the functions from a ring to itself and the set of all such functions forms a vector space over a field. To show that the set of all derivations on \(\mathbb{M}\) forms a subspace of \(\mathcal{C}(\mathbb{M})\) and therefore itself forms a vector space, note that there exists a null derivation \(= 0\) that acts as the additive identity and that the set is closed under vector addition and scalar multiplication:

\[
(X + Y)(fg) = X(fg) + Y(fg) = X(f)g + fX(g) + Y(f)g + fY(g)
\]

\[
= (X + Y)(f)g + f(X + Y)(g) = hX(f)g + fhX(g)
\]

7.1 A vector field on \(\mathbb{S}^1\) is a derivation on \(\mathcal{C}(\mathbb{S}^1)\). Since \(f(\theta)\) is a function on \(\mathbb{S}^1\) and \(f(\theta)\frac{d}{d\theta}\) is a derivation \(\forall f(\theta) \in \mathcal{C}(\mathbb{S}^1)\), \(f(\theta)\frac{d}{d\theta}\) is a vector field on \(\mathbb{S}^1\).

7.2 Neither product is itself a derivation because of the cross terms.

\[
UV(fg) = U(V(fg)) = U(V(f)g + fV(g)) = UV(f)g + V(U(g)) + U(f)V(g) + fUV(g)
\]

\[
\neq UV(f)g + fUV(g)
\]

However, in the commutator, these cross terms cancel out. Therefore, the commutator turns the space of derivations into an algebra.

\[
[U, V](fg) = UV(f)g + V(f)U(g) + U(f)V(g) + fUV(g)
\]

\[
- VU(f)g - U(f)V(g) - V(f)U(g) - fVU(g)
\]

\[
= [U, V](f)g + f[U, V](g)
\]

7.3 The complex Fourier series provides a convenient basis:

\[
L_m = \exp(i m \theta) \frac{d}{d\theta}
\]

The Lie bracket between two arbitrary basis elements is then given by:

\[
[L_m, L_n] = i(n - m)L_{m+n}
\]
Choosing the standard angular coordinate, the potential energy of the simple pendulum in natural units $\{m, g, l\} = 1$ is:

$$V = - \cos \theta$$

The points of equilibrium are the points at which $\theta$ is stationary under infinitesimal variations:

$$\frac{dV}{d\theta}(\theta_0) = 0$$
$$\sin \theta_0 = 0$$
$$\theta_0 = \{0, \pi\}$$

The second derivative matrix at each of these points has a single element and is then:

$$K = \partial^2 V(\theta_0) = \cos \theta_0$$

$$\begin{cases} 
K = 1 & \text{if } \theta_0 = 0 \\
K = -1 & \text{if } \theta_0 = \pi 
\end{cases}$$

Clearly, from the signs of the determinants of $K$, the $\theta_0 = 0$ stationary point is stable while that at $\theta_0 = \pi$ is unstable. The canonical momentum is the usual angular momentum:

$$p = \frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}$$

The linearized equations of motion are then:

$$\begin{cases} 
\dot{\theta} = p, \dot{p} = -\theta & \text{if } \theta_0 = 0 \\
\dot{\theta} = p, \dot{p} = \theta - \pi & \text{if } \theta_0 = \pi 
\end{cases}$$

To solve these, use an ansatz:

$$\theta(t) \equiv \theta_0 + A \exp(\lambda t)$$

$$\begin{cases} 
\lambda A \exp(\lambda t) = p, \dot{p} = -A \exp(\lambda t) & \text{if } \theta_0 = 0 \\
\lambda A \exp(\lambda t) = p, \dot{p} = A \exp(\lambda t) & \text{if } \theta_0 = \pi 
\end{cases}$$
Combining these two first order ODEs into one second order ODE gives:

\[
\begin{cases}
\{ \lambda^2 A = -A \} & \text{if } \theta_0 = 0 \\
\{ \lambda^2 A = A \} & \text{if } \theta_0 = \pi
\end{cases}
\]

Solving this eigenvalue equation then gives:

\[
\begin{cases}
\{ \lambda^2 + 1 = 0 \} & \text{if } \theta_0 = 0 \\
\{ \lambda^2 - 1 = 0 \} & \text{if } \theta_0 = \pi
\end{cases}
\]

Restoring parameters gives this as the eigenvalue:

\[
\begin{cases}
\{ \lambda = i \sqrt{\frac{\pi}{4}} \} & \text{if } \theta_0 = 0 \\
\{ \lambda = \sqrt{\frac{\pi}{4}} \} & \text{if } \theta_0 = \pi
\end{cases}
\]

Substituting these values for the eigenvalue back into the eigenvalue equation gives the eigenvector:

\[
\begin{cases}
\{ -A = -A \} & \text{if } \theta_0 = 0 \\
\{ A = A \} & \text{if } \theta_0 = \pi
\end{cases}
\]

Therefore, any nonzero one-dimensional vector \( A \) is an eigenvector. It is convenient and conventional to choose the normalized value \( A = 1 \) or \( A = \frac{1}{2} \). The final analytical solution must be real-valued which is achieved by taking an appropriate linear combination of the possibilities for the eigenvalue with the appropriately normalized eigenvector. The initial conditions determine the value of the constant \( \alpha \).

\[
\begin{cases}
\theta(t) = \alpha \exp \left( i \sqrt{\frac{\pi}{4}} t \right) + \alpha^* \exp \left( -i \sqrt{\frac{\pi}{4}} t \right) & \text{if } \theta_0 = 0 \\
\theta(t) = \alpha \exp \left( \sqrt{\frac{\pi}{4}} t \right) + \alpha^* \exp \left( -\sqrt{\frac{\pi}{4}} t \right) & \text{if } \theta_0 = \pi
\end{cases}
\]

9 Referencing class, the equations of motion are given by:

\[
\begin{aligned}
\dot{x}^i &= v^i, \\
\dot{v}^i &= -\frac{e}{m} K_{ij} x^j + \frac{e}{mc} B_{ij} v^j \}
\end{aligned}
\]

\[
K_{ij} = \partial_i \partial_j V(0)
= \begin{bmatrix}
-\omega & 0 & 0 \\
0 & -\omega & 0 \\
0 & 0 & 2\omega
\end{bmatrix}
\]

\[
B_{ij} = \begin{bmatrix}
0 & B & 0 \\
-B & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Applying the aforementioned ansatz gives:

\[ \lambda^2 A = -\frac{e}{m} KA + \lambda \frac{e}{mc} BA \]

This gives the eigenvalue equation

\[ \left( \lambda^2 + \frac{e}{m} K - \lambda \frac{e}{mc} B \right) A = 0 \]
\[ \det \left( \lambda^2 + \frac{e}{m} K - \lambda \frac{e}{mc} B \right) = 0 \]
\[ \det \begin{bmatrix} \lambda^2 - \frac{e}{m} \omega & -\lambda \frac{e}{mc} B & 0 \\ \lambda \frac{e}{mc} B & \lambda^2 - \frac{e}{m} \omega & 0 \\ 0 & 0 & \lambda^2 + 2 \frac{e}{m} \omega \end{bmatrix} = 0 \]

This is a cubic equation in \( \lambda^2 \). For greater transparency and ease, since the matrix is in block-diagonal form, the subdeterminants may be separately equated to zero. This gives the equation for the \( z \) eigenvalue as:

\[ \lambda^2 + 2 \frac{e}{m} \omega = 0 \]
\[ \lambda = i \sqrt{2 \frac{e}{m} \omega} \]

This confirms what was shown in class: if \( 2 \frac{e}{m} \omega > 0 \), which effectively means that the charge and the magnetic constant \( \omega \) have the same sign, the particle is stable in the \( z \) direction. Its angular oscillation frequency is:

\[ \nu = \sqrt{2 \frac{e}{m} \omega} \]

This leaves the equation for the remaining eigenvalues:

\[ \left( \lambda^2 - \frac{e}{m} \omega \right)^2 + \left( \lambda \frac{e}{mc} B \right) = 0 \]
\[ \left( \frac{e}{m} \omega \right)^2 + \left( \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega \right) \lambda^2 + \lambda^4 = 0 \]
\[ \lambda^2 = -\frac{1}{2} \left( \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega \right) \pm \frac{1}{2} \sqrt{ \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega }^2 - 4 \left( \frac{e}{m} \omega \right)^2 \]
\[ \lambda = i \left[ \frac{1}{2} \left( \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega \right) \pm \frac{1}{2} \sqrt{ \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega }^2 - 4 \left( \frac{e}{m} \omega \right)^2 \right] \]
\[ \nu = \left[ \frac{1}{2} \left( \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega \right) \pm \frac{1}{2} \sqrt{ \left( \frac{e}{mc} B \right)^2 - 2 \frac{e}{m} \omega }^2 - 4 \left( \frac{e}{m} \omega \right)^2 \right] \]
As shown in class, the stability condition for this submatrix $K'$ is:

$$\left( \frac{e}{c} B \right)^2 > \frac{2}{m} \frac{e}{m} \sqrt{\det K'} - \text{tr} K'$$

$$\det K' = \omega^2$$

$$\text{tr} K' = -2\omega$$

Solving for $B$ then gives:

$$B > \frac{c}{|e|} \sqrt{\frac{2}{m} \frac{e}{m} \omega + 2\omega}$$
Note: For each one-dimensional problem, interpret "orbits" as those in phase space.

Origin: \( \theta = 0 \), \( p = 0 \)

Intersection of axes: \( \theta = 0, p = 0 \)

(Same as \( \Theta \) in \( \Theta = 0 \))

Origin: \( z = 0 \), \( p_z = 0 \)

\( z = x \) or \( y \)

Note: Scales on each sketch will in general be quite different from one another depending on the values chosen for the parameters.
We have the Poincaré metric on the upper half-plane, given by
\[ ds^2 = \frac{dz^2 + d\bar{z}^2}{z \bar{z}} \]

If we write \( z = x + iy \),
\[ dz = dx + i dy \]
\[ d\bar{z} = dx - i dy \]
and \( z - \bar{z} = \text{Im} \{ z \} = y \).

Thus the Poincaré metric becomes
\[ ds^2 = -4 \frac{dz \, d\bar{z}}{(z - \bar{z})^2} \]

Consider the transformation \( z \rightarrow a \bar{z} + b = \frac{z}{cz + d} \).

Clearly, this means \( \bar{z} \rightarrow a \bar{z} + b = \frac{\bar{z}}{cz + d} \).

Now
\[ d\bar{z} = \frac{a(dz) - (a \bar{z} + b) \bar{c}(d\bar{z})}{(cz + d)^2} = \frac{(ad - bc) \, d\bar{z}}{(cz + d)^2} \]

Similarly, we must have
\[ d\bar{z}^* = \frac{(ad - bc) \, d\bar{z}^*}{(cz^* + d)^2} \]

Hence we obtain
\[ ds^2 = -4 \frac{(ad - bc)^2 \, d\bar{z} \, d\bar{z}^*}{(cz + d)^2 (cz^* + d)^2} \]

\[ = -4 \frac{(ad - bc)^2 \, d\bar{z} \, d\bar{z}^*}{(a \bar{z} + b)(cz + d) - (a \bar{z}^* + b)(cz^* + d)} \]

(cancelling common terms)
\[ = -4 \frac{(ad - bc)^2 \, d\bar{z} \, d\bar{z}^*}{(z - \bar{z})} \]

\[ = -4 \frac{d\bar{z} \, d\bar{z}^*}{z - \bar{z}} \]

(The last cancellation is valid as \( ad - bc \neq 0 \))
We may thus write
\[ \frac{d\hat{z}^2 + d\hat{y}^2}{\hat{z}^2 - \hat{z}^*\hat{y}^2} = \frac{-4dz\,dz^*}{(z - z^*)^2} = \frac{-4dz\,dz^*}{(z - z^*)^2} \]

or in other words, the Poincaré metric is invariant under the given transformation. Note that the transformation is an action of \( SL_2(\mathbb{R}) \) on \( \mathbb{C}^+ \), which I believe is a conformal mapping.

(1) We wish to determine the geodesic between \((x,y), (x',y')\), given the Poincaré metric above. The metric tensor is

\[
g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}
\]

During the process of obtaining the geodesic equation (which would in principle give us the metric) we derived the Euler-Lagrange (EL) equations (the usual equation of variational calculus) which are in this case:

\[
d \left( g^{ij} \frac{dx^i}{ds} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0
\]

Using \( x' = x \), \( x'^2 = y \), we have
\[
\frac{\partial g_{ij}}{\partial x} = 0 \text{ for all } i,j
\]

We thus obtain the equations

\[
\frac{d}{ds} \left( \frac{1}{y^2} \frac{dx^i}{ds} \right) = 0 \quad \frac{d}{ds} \left( \frac{1}{y^2} \frac{dy^i}{ds} \right) = \frac{1}{y^3} \left[ \frac{(dx^i)^2}{ds} + \frac{(dy^i)^2}{ds} \right] = 0 \quad (1)
\]

We thus obtain
\[ \frac{dx^i}{ds} = a \frac{y^i}{y^3} \text{ where } a \text{ is a real constant} \quad (2)
\]

Notice also that we have an 'invariant equation'

\[
1 = g^{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \frac{1}{y^2} \left[ \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 \right] \quad (3)
\]
Using this result, we have from (2) the result
\[ ds = \pm \frac{dy}{y \sqrt{1 - \frac{y^2}{a^2}}} \]
i.e. choosing to absorb integration constants on the left, and allowing \( s \) to represent the geodesic distance we find
\[ s + C = \int \frac{dz}{\sqrt{z^2 - \gamma a^2}} \quad (z = \frac{y}{a}, dz = \frac{1}{a} dy) \]
\[ = \cosh^{-1} (a z) \]
\[ \Rightarrow a z = \cosh (\pm (s - C)) = \cosh (s - C) \]
or, in other words
\[ y(s) = \frac{1}{z} = a \text{sech}(s + C). \]
When \( s = 0 \), \( y(0) = y \)
\[ \Rightarrow y = a \text{sech}(C) \Rightarrow a = y \cosh C, \]
i.e. 
\[ y(s) = y \cosh(C) \text{sech}(s + C) \quad \cdots (4) \]
Now \[ x = \frac{1}{a} \int y(s)^2 ds \]
\[ \Rightarrow x(s) = y \cosh(C) \tanh(S + C) + D. \]
When \( s = 0 \), \( x(0) = x \) and thus
\[ x = y \cosh C \tanh C + D \Rightarrow D = x - y \sinh C \]
Thus \[ x(s) = y \cosh(C) \tanh(C) + x - y \sinh(C) \quad \cdots (5) \]
Rearranging (4) and (5), squaring and adding and using \( \cosh^2 A + \sinh^2 A = 1 \) gives us
\[
(x(s) - x + y \sinh c)^2 + y(s)^2 = y^2 \cosh^2 c \quad \cdots (6)
\]
The remaining constant is found by requiring that \((x', y')\) lie on (6)
\[
(x' - x + y \sinh c)^2 + y'^2 = y'^2 \cosh^2 c.
\]
\[\text{i.e.}\]
\[
(x' - x)^2 + 2(x - x') y \sinh c + y^2 \sinh^2 c + y'^2 = y'^2 \cosh^2 c
\]
Rearranging we find that (in any case except \(x = x'\))
\[
\sinh c = \frac{y^2 - y'^2 - (x - x')^2}{2 y (x - x')^2}
\]
\[
\cosh^2 c = 1 + \sinh^2 c
\]
\[
= \frac{4 y^2 (x - x')^4 + (y^2 - y'^2 - (x - x')^2)^2}{(2 y (x' - x))^2}
\]
Thus the equations become
\[
[x(s) - x + \left\{ \frac{y^2 - y'^2 - (x - x')^2}{2 (x' - x)} \right\}]^2 + y(s)^2 = \frac{4(x - x')^4 y^4 + [y^2 - y'^2 - (x - x')^2]^2}{2 (x' - x)^4}
\]
This is clearly the equation of a circle that intersects the real line orthogonally (the center is on the real line, and hence the real line is a diameter, and diameters intersect the circle orthogonally).
In the limit that \(x' \to x\), the circle is centered
infinitely far away and has infinite radius, i.e. it approaches a straight line from \((x,y)\) to \((x',y')\) with equation \(x(s) = x'\).

\[
\begin{align*}
(x,y) & \quad \text{(limit as } x \to x') \\
(x',y') & \quad \\
(x,y') & \quad \\
(x',y) & \quad \\
(x,y) & \quad \\
(x',y) & \quad \\
(x,y') & \quad \\
(x',y') & \quad \\
\end{align*}
\]

The geodetic distance is found from (4). If we let \(\xi\) denote the geodetic distance we have

\[
y' = y \cosh(c) \text{sech}(\xi + c)
\]

\[
\Rightarrow \quad \xi = \text{sech}^{-1} \left( \frac{y'}{y} \text{sech}(c) \right) - c
\]

\[
= \text{sech}^{-1} \left( \frac{y'}{y} \left( \cosh^{-1}(c) \right) \right) - c
\]

\[
= \text{sech}^{-1} \left( \frac{2y' (x' - x)^2}{\sqrt{4y^2 (x'-x)^4 + (y^2 - y'^2 - (x'x)^2)^2}} \right) - \sinh^{-1} \left( \frac{y^2 - y'^2 - (x'x)^2}{2y (x'-x)^2} \right)
\]