Three-dimensional random $XY$ model: Application to the superfluid transition of $^4$He in porous media

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We study a three-dimensional classical $XY$ model with strongly fluctuating, spatially uncorrelated bond disorder. Finite-size scaling analysis of Monte Carlo simulation data has been used to determine critical exponents. Evidence for a possible shift in exponents from the pure $XY$ case is found. Connection to recent experiments on the superfluid phase transition of $^4$He in porous media is discussed.

The superfluid phase transition of liquid $^4$He in porous structures has been a topic of long-standing interest.\textsuperscript{1,2,3} Much effort has been devoted to understanding the onset of superfluidity at $T=0$ as a function of increasing helium density.\textsuperscript{2,3} Another problem has been the critical behavior at the very low transition temperatures seen in low-density thin films, which has been described in terms of a crossover to ideal-Bose-gas behavior.\textsuperscript{4,5} At the higher transition temperatures seen in saturated pores, the nature of the superfluid transition has also been extensively studied experimentally.\textsuperscript{1,5} In Vycor glass, a suppression of the normalized superfluid density $\rho_s(T)/\rho_s(0)$ as compared to bulk, is observed as $T_c$ is approached from below, but the critical exponents appear to be the same as bulk.\textsuperscript{1,5} Recent experiments in silica gels,\textsuperscript{5} however, have reported different critical exponents for $\rho_s$, suggesting that the random host may shift the transition to a new universality class.

Theoretically, the superfluid transition in this high-temperature case of saturated pores may be modeled by the classical $XY$ model. The complex order parameter of the superfluid transition has the same symmetry as the planar spins of the $XY$ model. Only a few attempts\textsuperscript{6,7} have been made to model the behavior of the superfluid density in this case. Modeling the porous media as a set of interconnected one-dimensional chains, Fishman and Ziman\textsuperscript{6} studied a regular bond-decorated spherical model. Garg et al.\textsuperscript{7} carried out simulations on bond and site diluted $3D XY$ models, with dilutions as large as 55%. In such spin models, the helicity modulus $\gamma(T)$ (spin-wave stiffness constant) plays the role of the superfluid density. As $T\to T_c$ from below, $\gamma$ vanishes as $\gamma \sim |t|^\nu$, with $\nu$ equivalent to the superfluid density exponent $\xi$. In both the preceding models, no substantial change in $\gamma(T)/\gamma(0)$ (apart from a linear rescaling of the temperature) was observed when compared with that of a regular periodic lattice, for all temperatures from $T_c$ down to $T=0$. Thus these models did not observe the suppression in $\rho_s(T)/\rho_s(0)$ near $T_c$ characteristic of helium in porous media, nor any changes in critical behavior. The critical behavior of the $3D XY$ model is expected by the Harris criterion\textsuperscript{8} to be unaffected by weak amounts of uncorrelated bond disorder. Weinrib and Halperin\textsuperscript{9} introduced a model in which algebraically correlated disorder may affect critical exponents. However, no explicit model calculation was carried out for helium. In this paper we study a new random-bond-decorated $3D XY$ model. The disorder we introduce, motivated by a simple physical model of a porous media, is uncorrelated but very strongly fluctuating. We observe for the first time a clear suppression of the normalized helicity modulus near $T_c$ from that of the regular periodic case, consistent with results on helium. A finite-size scaling analysis of Monte Carlo simulations is carried out to compute critical exponents, and we find results suggestive of a new universality class.

Our model is as follows. First, consider a regular $3D$ cubic lattice with one planar ($XY$) spin per node. Randomness is introduced by "decorating" each bond with a random number $n$ of planar spins. $n$ is taken from the probability distribution

$$p(n) = A e^{-n/l_0},$$

where $l_0$ sets the average number of spins on each bond and $A$ is the normalization factor. All spins on this decorated lattice interact through nearest-neighbor couplings. The Hamiltonian of this model is given by

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sum_{i,j=1}^{n_{ij}} \sigma_{ij} \sigma_{i+1,j+1} + s_i \sigma_1 + s_{n_{ij}} \cdot s_j,$$

where $s_i$ is the spin at node $i$ of the lattice, and $\{ \sigma_{ij} \}$ are the $n_{ij}$ spins on the bond-connecting nodes $i$ and $j$. As usual, the sum $\sum_{\langle ij \rangle}$ is over all nearest-neighbor nodes of the cubic lattice.

The model (2) resembles some key features of superfluid $^4$He in silica xerogel. For the xerogel in Ref. 5, the distribution of the pore diameters is sharply peaked at $d_p \approx 10$ nm which is much smaller than the coherence length of the superfluid $^4$He in it, $\xi \sim 10^2$ nm near $T_c$. The spatial variation of the phase of the macroscopic
wave function may be neglected over length scales much smaller than the coherence length \( \xi \). Therefore, one may treat the superfluid \(^4\)He in one pore as a one-dimensional system and model it by a linear chain of planar spins. We simplify the very complicated topology of highly inter-connected pores by modeling it as the connection of bonds on a 3D simple cubic lattice. The distribution (1) which we use, is suggested by the broad range of length scales believed appropriate for gels (Vycor in contrast, seems characterized by a narrower length distribution).\(^{5,10}\)

Before numerically simulating the model (2), we first integrate over the degrees of freedom of the bond spins \( \{ \sigma_{ij} \} \) in the spirit of the real-space normalization group. A strict decimation over bond spins would result in an effective node-node interaction which is no longer a cosine. We simplify this by replacing the decimated bond spins by a cosine interaction with a coupling \( K_{ij} \) determined as follows: We require \( \langle S_0 \cdot S_{n+1} \rangle_K \), the spin-spin correlation function of the chain with \( n \) intermediate spins at coupling \( K = J/T \), to be equal to that of a two-spin chain with coupling \( K_{ij} \), i.e., \( \langle S_0 \cdot S_1 \rangle_{K_{ij}} \). These \( K_{ij} \) are given in terms of the number of bond spins \( n_{ij} \) by solving\(^{11}\)

\[
I_1(K_{ij})/I_0(K_{ij}) = [I_1(J/T)/I_0(J/T)]^{n_{ij}+1}.
\]

(3)

\( I_n \) is the \( n \)-th order modified Bessel function. Thus we reduce the decorated-bond problem to a random-bond \( XY \) model

\[
\mathcal{H}_{\text{eff}} / T = - \sum_{(ij)} K_{ij} \cos(\theta_i - \theta_j),
\]

(4)

where \( \theta_i \) is the angle of the node spin \( s_i \). An interesting feature of the bond distribution generated by this procedure is that it is algebraic at small \( K \). From Eqs. (1) and (3) we find at small \( K \) that the probability density to have a bond \( K_{ij} = K \) is given by

\[
\gamma_K = \frac{T}{N^2} \left( \left( \sum_{(ij)} K_{ij} \cos(\theta_i - \theta_j) \hat{\mu}_j \right)^2 - \left( \sum_{(ij)} K_{ij} \sin(\theta_i - \theta_j) \hat{\sigma}_j \hat{\mu}_j \right)^2 \right),
\]

(7)

We average \( \gamma_K \) over \( \hat{\mu} = \hat{x}, \hat{y}, \text{ and } \hat{z} \) to achieve better statistics. In Fig. 1 we show our results for normalized helicity modulus versus \( T / T_c \), for the size \( L = 10 \). We show the pure \( XY \) model and our random-bond model with distributions (1) given by \( I_0 = 3, 5, \text{ and } 10 \). We see clearly the suppression of \( \mathcal{Y}(T)/\mathcal{Y}(0) \) near \( T_c \) for the random case as compared to the pure case.

To analyze critical behavior, the results of the helicity modulus \( \mathcal{Y} \) and order parameter \( |M|^2 \) have been fitted to second-order expansions of the finite-size scaling functions,\(^{14}\)

\[
\mathcal{Y}(K,L) = L^{-v/\nu}[H_0 + H_1 L^{1/\nu}(K - K_c) + H_2 L^{2/\nu}(K - K_c)^2],
\]

(8)

\[
|M(K,L)|^2 = L^{-\beta/\nu}[\Phi_0 + \Phi_1 L^{1/\nu}(K - K_c) + \Phi_2 L^{2/\nu}(K - K_c)^2],
\]

(9)

where \( K = J/T \). In our fits to \( \mathcal{Y} \), we have assumed the Josephson scaling relation \( v = (d - 2)\nu \) (\( \nu = \nu \) in \( d = 3 )^{15}\). Our fitting procedure has been described in detail elsewhere.\(^{12}\)

Simulation of the model (4) has been done for bond distributions (1) characterized by \( I_0 = 3, 5, \text{ and } 10 \). For \( I_0 = 5 \), even sizes \( L = 6-14 \) were simulated. To test that we are in the scaling regime, we drop data from the successively lowest values of \( L \) until the fitted parameters remain unchanged. We have used only the data from \( L = 10, 12, \text{ and } 14 \) for the final fits. For \( I_0 = 3 \) and 10, only sizes \( L = 10, 12, \text{ and } 14 \) were simulated. The fitted curves for \( \mathcal{Y} \) and \( |M|^2 \), for the case \( I_0 = 5 \), are shown in Fig. 2. The fitted parameters are shown in Table I. Results from the helicity modulus and order parameter fittings agree within statistical error. While exponents for the case \( I_0 = 3 \) agree with those of the pure \( XY \) model, the exponent \( \nu \) is seen to increase from the pure \( XY \) value.
as \( l_0 \) increases. Our values compare with the exponents 
\( \xi \approx 0.8 - 0.9 \) found in experiments \(^5\) on \(^4\)He in silica gels 
\( (\xi \equiv \nu, \nu = \nu \text{ in } d = 3) \).

The effects of uncorrelated bond disorder on critical behavior is generally discussed in terms of the Harris criterion. \(^3\) This argues for the irrelevance of weak disorder if the specific-heat exponent \( \alpha < 0 \), as is the case for the pure 3D XY model where \( \alpha = -0.007 \pm 0.006 \). \(^6\) For our random bond XY model, however, disorder is strong. In Table I we show the value of the bond strength fluctuation, \( \langle (\Delta K_{ij})^2 \rangle_c / \langle K_{ij} \rangle_c^2 \), at the critical temperature. We also show the value of the exponent \( a(T_c) \) of the bond probability distribution (5) at small \( K \). As one can see, the fluctuation is stronger and \( P_T(K) \) is more divergent at small \( K \) as \( l_0 \) gets larger. These observations suggest that the Harris criterion may not be applicable to such a strongly disordered system. A random Heisenberg model

![Diagram](image)

**FIG. 1.** Normalized helicity modulus \( \gamma(T) / \gamma(0) \) vs \( T / T_c \) for the pure 3D XY model (Ref. 12) and our random model with the bond distribution (1) characterized by \( l_0 = 3, 5, \) and 10. Suppression of \( \gamma(T) / \gamma(0) \) near \( T_c \) is clearly observed in the random case. Data is for a lattice of size \( L = 10 \). Inset shows data on finer scale.

![Diagram](image)

**FIG. 2.** The finite-size scaling behavior (a) of the helicity modulus \( \gamma(T, L) \), and (b) of the order parameter \( |M(T, L)|^2 \) for the bond distribution characterized by \( l_0 = 5 \). Symbols with error bars represent the Monte Carlo simulation results. The solid lines are the results of the fits to Eqs. (8) and (9) using data from sizes \( L = 10 - 14 \). In (a), \( \nu = 0.74 \) is used in making the horizontal axis of the plot. In (b), \( \nu = 0.72 \) and \( \beta = 0.43 \) are used in making the axes of the plot. The values of the fitted parameters are shown in Table I. \( T_c \equiv 1 / K_c \approx 0.613 \).

With a similar bond strength distribution diverging algebraically at \( K \to 0 \) has been studied at the percolation threshold. \(^7\) There it is found that both critical exponent \( t \) describing the vanishing of helicity modulus at the percolation threshold and the thermal-percolative crossover exponent \( \phi \) vary with the power of the algebraic distribution.

To conclude, we have proposed a physical model for the superfluid transition of \(^4\)He in porous media, which we reduce to a random-bond 3D XY model [Eq. (4)]. Disorder is strong and the Harris criterion may not apply. We have carried out Monte Carlo simulation of the model, and used finite-size scaling to find the critical exponents. We find that the normalized helicity modulus (superfluid density) is suppressed near \( T_c \) from the pure (bulk) case, as is observed experimentally. For the more

![Table](image)

**TABLE I.** Table of simulation results for different bond distributions. \( \gamma^{(T)} \) and \( K_c^{(T)} \) are the thermal exponent and the critical coupling as determined by fits to helicity modulus \( \gamma \), Eq. (8). \( \gamma^{(M)} \), \( K_c^{(M)} \), and order parameter exponent \( \beta \) are determined by fits to \( |M|^2 \), Eq. (9). \( \gamma^{(T)}, \gamma^{(M)} \) and \( K_c^{(T)}, K_c^{(M)} \) agree within statistical error. \( \langle (\Delta K_{ij})^2 \rangle_c / \langle K_{ij} \rangle_c^2 \) is the fluctuation in random-bond strength at \( T_c \). \( a(T_c) \) is the exponent of the small-\( K \) bond distribution, Eq. (5), at \( T_c \). Results for the pure XY model come from Ref. 12.

<table>
<thead>
<tr>
<th>( l_0 )</th>
<th>( \gamma^{(T)} )</th>
<th>( \gamma^{(M)} )</th>
<th>( K_c^{(T)} )</th>
<th>( K_c^{(M)} )</th>
<th>( \beta )</th>
<th>( \langle (\Delta K_{ij})^2 \rangle_c / \langle K_{ij} \rangle_c^2 )</th>
<th>( a(T_c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure</td>
<td>0.68±0.02</td>
<td>0.67±0.02</td>
<td>0.453±0.0006</td>
<td>0.453±0.0006</td>
<td>0.36±0.01</td>
<td>0.86</td>
<td>0.50</td>
</tr>
<tr>
<td>3</td>
<td>0.70±0.05</td>
<td>0.65±0.04</td>
<td>1.199±0.006</td>
<td>1.200±0.006</td>
<td>0.37±0.04</td>
<td>1.1</td>
<td>0.57</td>
</tr>
<tr>
<td>5</td>
<td>0.74±0.04</td>
<td>0.72±0.03</td>
<td>1.63±0.01</td>
<td>1.61±0.01</td>
<td>0.43±0.04</td>
<td>1.1</td>
<td>0.57</td>
</tr>
<tr>
<td>10</td>
<td>0.85±0.06</td>
<td>0.92±0.08</td>
<td>2.46±0.02</td>
<td>2.51±0.02</td>
<td>0.42±0.05</td>
<td>1.4</td>
<td>0.64</td>
</tr>
</tbody>
</table>
strongly disordered cases that we studied ($l_0 = 5, 10$), we find a helicity modulus exponent $v = v$ which is larger than the pure $XY$ value. This larger exponent is consistent with the experimental trend for helium in gels. While this is suggestive of a new “random” universality class, we cannot rule out the possibility that our simulations are still in a crossover regime, and that for larger system sizes $L$, the pure $XY$ values could be obtained.\[18\

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10D. D. Awschalom (private communication).
11Consider a chain of $n + 2$ planar spins with angles $\theta_i, i = 0$ to $n + 1$. The Hamiltonian for the chain is $\mathcal{H}/T = -K \sum_{i=1}^{n+1} \cos(\theta_i - \theta_{i+1})$. By transforming variables to angle differences, one may easily compute the correlation function for the end spins, $\langle \cos(\theta_0 - \theta_{n+1}) \rangle_K = [I_1(K')/I_0(K')]^{x+1}$. Similarly, for a two-spin chain we have $\langle \cos(\theta_0 - \theta_1) \rangle_K = I_1(K')/I_0(K')$. Equating the two results gives Eq. (3).
13This is a straightforward generalization to the random $XY$ model, of the formula used in S. Teitel and C. Jayaprakash, Phys. Rev. B 27, 598 (1983).
14We have also done fits to third-order expansions of the scaling functions. The results of these fits agree, within the statistical error, with the values in Table I.
15B. D. Josephson, Phys. Lett. 21, 608 (1966); M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A 8, 1258 (1973). We have independently checked the Josephson relation in the random model by doing fits to $\gamma$ where $v/v'$ is left as a free parameter. We find that $v/v' \approx 0.999 - 1.001$.
18To check the sensitivity of our fits to the values of the critical exponents, we have done fits to our data in which $\nu$ and $\beta$ were fixed at the values for the pure $XY$ model. We found these fits increased the $\chi^2$ by a factor $\sim 2$ over those where $\nu$ and $\beta$ were left as free parameters.