Vortex-line fluctuations in model high-temperature superconductors

Ying-Hong Li

Institute for Theoretical Physics, University of Utrecht, 3508 TA Utrecht, The Netherlands

S. Teitel

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

(Received 15 June 1992)

We carry out Monte Carlo simulations of the uniformly frustrated three-dimensional XY model, as a model for vortex-line fluctuations in a high-$T_c$ superconductor in an external magnetic field. A density of vortex lines of $f = \frac{1}{25}$ is considered. We find two sharp phase transitions. The low-$T$ superconducting phase is an ordered vortex-line lattice. The high-$T$ normal phase is a vortex-line liquid, with much entangling, cutting, and loop excitations. An intermediate phase is found, which is characterized as a vortex-line liquid of disentangled, approximately straight, lines. In this phase, the system displays superconducting properties in the direction parallel to the magnetic field, but normal behavior in planes perpendicular to the field. A detailed analysis of the vortex structure function is carried out.

I. INTRODUCTION

While an understanding of the microscopic mechanism leading to high-temperature superconductivity has remained elusive, much recent research has focused on obtaining a phenomenological understanding of the behavior of these materials in the presence of applied currents and magnetic fields. Such treatments have been based on the familiar Landau-Ginzburg approach for type-II superconductors, which assumes only that macroscopically, the superconductivity is adequately described by a complex order parameter, as is the case for classical BCS materials. In this context, the high-$T_c$ materials are generally believed to differ from the classical superconductors in that fluctuation effects are dramatically enhanced. $\lambda$, $\kappa$, and ratio of magnetic penetration to coherence lengths, $\kappa = \lambda/\xi$, in these materials lead to a significantly larger critical region in temperature, about $T_c$.

In the presence of an applied magnetic field $H_{cl} < H < H_{c2}$, the low-temperature phase of a pure system is the Abrikosov vortex-line lattice. As temperature increases, it has been argued that fluctuations will cause this vortex-line lattice to melt at a temperature well below that given by the mean field $H_{c2}(T)$ line, leading to a vortex-line liquid phase. Experimental evidence for such a scenario has been cited in mechanical oscillator measurements, as well as by the observation that the onset of sizable reversible magnetization generally occurs at a temperature well above that where resistivity vanishes, an indication of strong fluctuation effects. Several theoretical works have sought to describe the nature of this vortex-line lattice melting transition, as well as predict properties of the vortex-line liquid phase. One commonly used approach is an effective elastic theory, in which the energy of transverse fluctuations of the vortex lines from their lattice positions is computed in terms of nonlocal elastic constants determined from the mean field Landau-Ginzburg theory. The Lindemann criterion is then used to estimate melting when such transverse fluctuations become a sizable fraction of the vortex-line spacing. Another theoretical model has been the “two-dimensional boson” approximation, in which vortex lines, fluctuating in a directed fashion along the direction of the applied magnetic field, are viewed as the “world lines” of two-dimensional (2D) interacting bosons. Many-body methods applied to the two-dimensional boson problem have then been used to estimate melting in the dilute line limit, as well as to compute properties of the vortex-line liquid. The above theoretical models suffer from several important limitations. The effective elastic theory loses validity once the vortex lines are in a liquid phase. The “2D boson” approximation assumes a simplified vortex-line interaction, that applies mainly when lines do not wander too much from parallel alignment with the applied magnetic field. Both models assume that the vortex lines thread the system in a directed fashion; each line passes through each plane perpendicular to B only once. Closed vortex-line loop excitations, which have been argued to play a significant role in the $B = 0$ case, as well as in the finite $B$ case are ignored. In view of this, it is useful to have a well-defined statistical model, for which numerical simulations can be carried out, and in which the effects of vortex-line fluctuations may be quantitatively examined.

In this paper we present the results of such a study, based on a simplified model which we believe captures the essential physics of a fluctuating Landau-Ginzburg superconductor; the uniformly frustrated three-dimensional
XY model. We have previously introduced this model in two earlier works.\textsuperscript{12,13} Here we provide details, as well as surprising results. We find that as the vortex-line lattice is heated, the system undergoes two distinct thermodynamic transitions. First there is a transition characterized by the loss of phase coherence in planes perpendicular to the applied magnetic field, while phase coherence is retained parallel to the field. In this intermediate region, the vortex-line lattice disorders in the transverse direction, while individual lines remain essentially straight and unentangled from each other. Then at a higher temperature, phase coherence is lost in the direction parallel to B as well. This high-temperature region is characterized by vortex-line wandering and many closed loop excitations. The vortex lines are highly entangled, yet they easily cut through one another.

The remainder of our paper is organized as follows. In Sec. II we present a detailed description of our model, and the approximations involved. In Sec. III we present the results of our Monte Carlo simulations: Sec. IIIA discusses computation of the helicity modulus, which measures the loss of phase coherence; Sec. III B discusses the nature of vortex-line fluctuations in the various thermodynamic phases; and Sec. III C presents a detailed analysis of the vortex structure function, which is used to extract effective elastic constants and correlation lengths. In Sec. IV we summarize our conclusions.

### II. DESCRIPTION OF THE MODEL

It is generally believed that the phenomenological behavior of the high-temperature superconductors is well described by the familiar Landau-Ginzburg free energy functional,\textsuperscript{14}

\begin{equation}
F[\psi(r), A(r)] = \int d^3 r \left\{ \alpha |\psi|^2 + \frac{1}{2} \beta |\psi|^4 + \frac{\hbar^2}{2m} \left( -i \nabla + \frac{2\pi}{\Phi_0} A\right) \psi \right|^2 + \frac{1}{8\pi} |\nabla \times A|^2 \right\},
\end{equation}

where $\psi$ is the superconducting order parameter, $A$ the magnetic vector potential, and $\Phi_0 = \hbar c/2e$ the flux quantum. The local magnetic induction is $b(r) = \nabla \times A(r)$. The first two terms represent the condensation energy, the third term is the kinetic energy of the flowing supercurrents, and the fourth term is the magnetic field energy. We choose to work throughout this paper with the Helmholtz free energy, which is a function of the average magnetic induction, $B = (1/\text{Vol}) \int d^3 r b(r)$, inside the bulk superconductor. The applied magnetic field $\mathbf{H}$ may in principle be determined by an appropriate thermodynamic derivative of the Helmholtz free energy, but its value is of no direct concern for our calculations. For simplicity, we consider here only a material with isotropic couplings, and indicate later how the effects of anisotropy may be included in our model.

The mean field solution, appropriate for low-temperature superconductors, is obtained by minimizing Eq. (1) with respect to both $\psi$ and $A$, subject to the constraint of constant $\mathbf{B}$. For $\mathbf{B} = 0$, one gets the uniform solution $\psi(r) = \sqrt{|\alpha|/\beta} \equiv \psi_0$ in the superconducting phase, $\alpha < 0$ ($\psi_0 = 0$ in the normal phase, $\alpha > 0$). For $0 < B < H$, in the mixed phase of a type-II superconductor, the result is the familiar Abrikosov flux tube lattice.\textsuperscript{14} Each flux tube carries one $\Phi_0$ of flux, and consists of a vortex line in the phase $\psi(r) = |\psi(r)| e^{i\theta(r)}$. The vortex line is surrounded by a normal region of radius the coherence length,

\begin{equation}
\xi_0 = \hbar/\sqrt{2m|\alpha|}
\end{equation}

and the magnetic field is confined to within a region of radius the magnetic penetration length,

\begin{equation}
\lambda = \sqrt{mc^2/16\pi e^2|\psi_0|^2}.
\end{equation}

The density of flux tubes is $B/\Phi_0$, hence the separation between vortex lines is

\begin{equation}
\alpha_v \simeq \sqrt{\Phi_0/B}.
\end{equation}

For the high-temperature superconductors, we now wish to compute thermodynamic quantities, averaging over all fluctuations of $\psi$ and $A$ (subject to the constraint of constant $\mathbf{B}$), weighting configurations with the Boltzmann factor $e^{-F[\psi, A]/k_B T}$. In order to carry out this average numerically, with Monte Carlo simulations, we make the following approximations.

For $\lambda \gg \alpha_v$, the magnetic fields of the individual flux tubes strongly overlap, leading to an approximately uniform local magnetic induction. In this limit, we therefore make the approximation of replacing the fluctuating $A$ with a fixed one, giving a spatially uniform magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$. We will discuss the consequences of this approximation on the effective vortex-line interaction in more detail below. Next we discretize\textsuperscript{15} our model by placing it on a periodic cubic mesh of lattice constant $a$, resulting in the free energy functional

\begin{equation}
F[\psi_{ij}, A_{ij}] = a^3 \sum_j \left\{ \left( \alpha + \frac{\hbar^2}{2ma^2} \right) |\psi_{ij}|^2 + \frac{1}{2} \beta |\psi_{ij}|^4 \right\} - \frac{ah^2}{2m} \sum_{\langle ij \rangle} \psi_{ij}^* e^{-iA_{ij}} \psi_i + c.c.,
\end{equation}

where we have dropped the now constant magnetic energy term. Here $i, j$ denote sites of the cubic mesh, $\langle ij \rangle$ denote nearest-neighbor bonds of the mesh, and

\begin{equation}
A_{ij} = \frac{2\pi}{\Phi_0} \int_i^j \mathbf{A} \cdot dl.
\end{equation}

Finally, we choose the lattice constant of the numerical mesh $a = \xi_0$, the coherence length of Eq. (2). Thus the decay to zero of the superconducting wave function $\psi$ at the normal core of a vortex line takes place within one unit cell of the mesh. Outside the normal core, we can make a London approximation and set $|\psi| = \psi_0$, a constant. Thus on the mesh, we consider only fluctuations...
of the form \( \psi_j = \psi_0 e^{i\theta_j} \), where the amplitude is fixed, and only the phases \( \theta_j \) vary. Details concerning the normal cores of vortex lines are treated approximately by the small length cutoff \( a = \zeta_0 \) imposed by the mesh. Applying this to the free energy in Eq. (5) results, within additive constants, in the effective Hamiltonian for a frustrated three-dimensional XY model,

\[
F[\theta_i] = -J_0 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - A_{ij}),
\]

(7)

where Eqs. (3) and (5), \( \Phi_0 = \hbar c/2e \), and \( \zeta_0 = a \) give

\[
J_0 = \frac{\Phi_0^2 \zeta_0}{16\pi^3 \lambda^2}.
\]

(8)

Anisotropy, or impurities, could now easily be incorporated into the model by letting the coupling \( J_0 \) vary appropriately on different bonds of the mesh. In this work we continue to consider only the case of isotropic couplings. We expect that the qualitative behaviors we find will be to a large extent the same as in anisotropic systems.\(^{12,16}\)

The Hamiltonian (7) above has been studied extensively in two dimensions as a model for Josephson junction arrays,\(^{17,18}\) and in two and three dimensions as a model for granular superconductors.\(^{19}\) Its application to behavior in the high-temperature superconductors is best realized in the limit \( \zeta_0 \ll a \ll \lambda \), where the above approximations should apply. This condition may be rewritten\(^{14}\) as \( H_{c1} \ll H \ll H_{c2} \). For the high-\( T_c \) materials, where \( \kappa = \lambda/\zeta_0 \sim 100 \) is large, this will cover a large range of experimentally accessible magnetic fields \( H \). For YBaCuO, for example, this condition holds for fields on the order of a few Tesla.

Much of the theoretical work concerning fluctuations in the high-\( T_c \) superconductors has been discussed in terms of fluctuations of the vortex lines induced by a finite \( B \). It is therefore of interest to consider the effective vortex-line interaction in our model (7). Towards this end, it is convenient to replace the cosine function of Eq. (7) with the periodic Gaussian or Villain function:\(^{20}\)

\[
F[\theta_i] = J_0 \sum_{\langle ij \rangle} \int V(\theta_i - \theta_j - A_{ij}),
\]

(9)

where

\[
V(\alpha) \equiv -\frac{1}{2} J_0 (\alpha - 2\pi m)^2/T \ln \left\{ \sum_{m=-\infty}^{\infty} \exp \left\{ -\frac{1}{2} J_0 (\alpha - 2\pi m)^2/T \right\} \right\}.
\]

(10)

This has the effect of eliminating the coupling between spin wave and vortex excitations of the phase variables \( \theta_i \) in the Hamiltonian (7).

Making a standard duality transformation\(^{21}\) on this new Hamiltonian (9) gives the effective Hamiltonian in terms of the vortex degrees of freedom:

\[
F_\nu[n_\mu(i)] = 2\pi^2 J_0 \sum_k \sum_{\mu} [n_\mu(i) - f_\mu(i)]
\]

\[
\times \int [n_\mu(j) - f_\mu(j)] G(r_i - r_j).
\]

(11)

Here \( n_\mu(i) \) is the integer vorticity in the phase \( \theta \) going around the plaquette with normal \( \mu = \hat{x}, \hat{y}, \hat{z} \) of the cubic unit cell centered at the dual mesh site \( i \), and \( f_\mu(i) \) is the number of flux quanta of magnetic induction through this plaquette,

\[
2\pi f_\mu(i) = \frac{2\pi}{\Phi_0} \oint A \cdot dl = A_{jk} + A_{kl} + A_{lm} + A_{mj},
\]

(12)

where the sum is taken counterclockwise over the bonds forming the edges of the plaquette. For our uniform magnetic field approximation, with \( B \) along the \( \hat{z} \) direction,

\[
f_z(i) = B \zeta_0^2 / \Phi_0 \equiv f, \quad f_x(i) = f_y(i) = 0.
\]

(13)

In this case, the ground state of Eq. (11) will consist of a periodic structure of straight vortex lines, aligned with \( B \) in the \( \hat{z} \) direction, with density \( f \).

The interaction potential \( G \) in Eq. (11), is the lattice Green’s function which solves

\[
D_\mu G(r_j - r_k) = -\delta_{i,k},
\]

(14)

where \( D_\mu \) is the lattice Laplacian. \( G(r) \) is most easily expressed in terms of its Fourier transform

\[
G(r) = \frac{1}{N} \sum_k G_k e^{i k \cdot r},
\]

(15)

where \( N \) is the number of sites in the mesh, and

\[
G_k = 1/K^2 = 1/(K_x^2 + K_y^2),
\]

(16)

\[
K_x^2 = 2 - 2 \cos k_x a - K_y^2 = 4 - 2 \cos k_x a - 2 \cos k_y a.
\]

Note \( K^2 \approx (ka)^2 \) for small \( ka \), however they are different at large \( ka \) due to the discreteness of the mesh. For large \( r \gg a \), \( G(r) \approx 1/(4\pi r) \).

We thus see that the interaction between vortex lines in our model is the Coulomb interaction, \( G_k = 1/K^2 \).

For comparison, the mean field London interaction in the continuum,\(^{22}\) obtained by minimizing the free energy Eq. (1) with respect to a spatially varying \( b(r) \), is

\[
G^{\text{London}}_k = 1/(k^2 + \lambda^2),
\]

which is a screened Coulomb interaction with screening length \( \lambda \). Thus our approximation of taking a uniform magnetic induction \( b(r) = B \), and ignoring spatial variations and fluctuations in \( b \), corresponds formally to making this screening length infinite. Our model therefore strongly suppresses long length scale (small \( k \)) vortex-line density fluctuations, as compared to the true superconductor. However we expect that the energetics of fluctuations on small length scales \( k^{-1} \approx \lambda \) are well approximated. Provided \( a \ll \lambda \), the model should therefore correctly describe the high \( k \) \( \sim 1/a \) fluctuations describing collisions between near-neighbor vortex-lines, which are believed responsible for the melting of the vortex line lattice.\(^{4}\)

It will be convenient for later to write Eq. (11) in terms of Fourier transforms. Defining

\[
n_\mu(k) \equiv \sum_i n_\mu(i) e^{-i k \cdot r_i},
\]

(17)
we have as the Hamiltonian
\[
F_v[n_\mu(i)] = \frac{2\pi^2 J_0}{N} \sum_{\mu,k \neq 0} n_\mu(k)n_\mu(-k)G_k.
\] (18)

The \( k = 0 \) contribution to \( F_v \) would be \( n_\mu(k = 0) - N f_{\mu k}^2 G_{k=0} \), where \( f_{\mu k} \) is as in Eq. (13). However as \( G_{k=0} \) diverges [see Eq. (16)], the requirement that the total free energy be finite ensures that we only have configurations where the average vortex density in direction \( \bar{\mu} \) is \( f_{\bar{\mu}} \); hence the \( k = 0 \) piece vanishes.

**III. MONTE CARLO SIMULATIONS**

To investigate the behavior of the model, we carry out Metropolis Monte Carlo simulations, of the Hamiltonian (9), on a cubic mesh with \( N = L_z L_x^2 \) sites and periodic boundary conditions in all directions. Our simulations are carried out directly in terms of the phase variables \( \theta_i \), with no additional presumptions concerning the behavior of vortex lines, beyond those implicit in Eq. (9). Simulations of this same model, only working directly in terms of the vortex variables and using the Hamiltonian (11), have recently been reported by Cavalcanti et al.\(^{2,3}\)

Previously we have reported\(^{12,13}\) on our simulations using a density of field induced vortex lines of \( f = \frac{1}{2} \). In this work we report on simulations using a density of lines \( f = \frac{1}{2} \), with an average spacing between lines of \( a_\nu = 5a_\nu \). This value of \( f \) was chosen so as to increase the ratio \( a_\nu / a \), and hence reduce effects due to the discreteness of our numerical mesh, while making \( f \) large enough so as to have a large number of lines contained within a system of computational size. Our present simulations are carried out for a mesh of size \( L_\perp = 50 \) and \( L_z = 24 \), giving 100 vortex lines in the ground state. The ground state of our system we believe to be the periodic extension of the unit cell of straight vortex lines, shown in Fig. 1. Although we have no way to prove that this is the true ground state, anesthylating with Monte Carlo simulations failed to result in any state with lower energy. That our presumed ground state is a square lattice of lines, instead of the triangular lattice expected in a continuum model, is due to the competition between vortex-line interactions and the cubic discreteness of our numerical mesh. Most of our calculations represent runs of 2000 Monte Carlo sweeps to equilibrate, with 20000 sweeps for computing averages. Each Monte Carlo sweep refers to one pass through the entire cubic mesh. Errors are estimated either by block averaging, or from comparing independent runs. Henceforth we measure length in units where \( \xi_0 = a = 1 \), and temperature in units where \( k_B = 1 \).

**A. Helicity modulus**

The most convenient method to detect phase coherence, i.e., superconductivity in the model Hamiltonian (9), is to compute the helicity modulus.\(^{24}\) The helicity modulus \( \Upsilon_\mu \) in the direction \( \hat{\mu} \) is defined as follows.\(^{25}\)

Imagine applying boundary conditions such that the phase across the system in the direction \( \hat{\mu} \) is twisted by an amount \( L_\mu \delta \). If \( \delta = \Phi \), for example, we would require
\[
\theta(x = L_x, y, z) - \theta(x = 0, y, z) = L_x \delta.
\] (19)

Now transform to the new set of variables
\[
\theta'_i = \theta_i - (r_i \cdot \hat{\mu}) \delta.
\] (20)

If \( \theta_i \) obey the twisted boundary conditions (19), then these new variables \( \theta'_i \) are easily seen to obey periodic boundary conditions. In terms of \( \theta'_i \), the system with the twisted boundary conditions is described by the Hamiltonian
\[
F[\theta'_i; \delta] = J_0 \sum_{\langle ij \rangle} V[\theta'_i - \theta'_j - A_{ij} - (\hat{e}_{ij} \cdot \hat{\mu}) \delta],
\] (21)

where \( \hat{e}_{ij} \) is the unit vector from site \( i \) to site \( j \). One can now form the partition function
\[
Z(\delta) = \sum_{\{\theta'_i\}} e^{-F[\theta'_i; \delta]/T},
\] (22)

where the sum is only over configurations \( \{\theta'_i\} \) satisfying periodic boundary conditions. The total free energy is then
\[
\mathcal{F}(\delta) = -T \ln Z(\delta)
\] (23)

and the helicity modulus is now defined as
\[
\Upsilon_\mu = \frac{1}{N} \frac{\partial^2 \mathcal{F}}{\partial \delta^2}
\] (24)

where \( N = L_z L_x^2 \). \( \Upsilon_\mu \) measures the stiffness to twisting the phase up in the \( \hat{\mu} \) direction, and may be thought of
as the renormalized coupling $J_0$ on infinitely long length scales. Equivalently, $T$ may be viewed as the effective superfluid density, or via Eq. (8), as determining the renormalized magnetic penetration length, $\sim 1/\sqrt{\xi_0}$. When $T_\mu > 0$ there is phase coherence in the $\hat{\mu}$ direction, and when $T_\mu = 0$, phase coherence is lost. The vanishing of $T_\mu$ in a particular direction therefore signals a superconducting to normal transition.

In terms of the definitions Eqs. (21)–(24), we can write the helicity modulus as the correlation function

$$T_\mu = \frac{J_0}{N} \left\langle \sum_{\ell j} V''(\theta_i - \theta_j - A_{ij}) (\hat{\ell} \cdot \hat{\mu})^2 \right\rangle - \frac{J^2}{T_N} \left\{ \left\langle \left[ \sum_{\ell j} V'(\theta_i - \theta_j - A_{ij})(\hat{\ell} \cdot \hat{\mu}) \right]^2 \right\rangle \right. $$

$$\left. - \left\langle \sum_{\ell j} V'(\theta_i - \theta_j - A_{ij})(\hat{\ell} \cdot \hat{\mu}) \right\rangle^2 \right\},$$

which can be evaluated directly in the Monte Carlo simulation. $V'$ and $V''$ are the first and second derivatives of the Villain function defined in Eq. (10).

In Fig. 2 we show our results for $T_x$ and $T_\perp \equiv \frac{1}{2}(T_x + T_y)$, as a function of temperature. Both heating from the ground state (Fig. 1) and cooling from random are shown. The results clearly indicate two sharp transitions. As $T$ is increased, there is first a transition $T_{c\perp} \simeq 1.3J_0$ at which $T_\perp \rightarrow 0$ but $T_x > 0$. Phase coherence is lost in the directions perpendicular to the applied magnetic field, however phase coherence persists in the direction parallel to the magnetic field. Then, at a higher $T_{ce} \simeq 2.5J_0$, $T_x \rightarrow 0$ and phase coherence is lost in the parallel direction as well, and we have the completely normal phase. In the intermediate phase, $T_{c\perp} < T < T_{ce}$.

B. Vortex-line fluctuations

We now seek to describe the nature of vortex-line fluctuations in the three distinct phases discussed above. We will find that the high-temperature phase is one in which vortex lines wander, entangle, and cut through each other easily. In this phase, we also find that the density of vortex lines is greatly increased by the excitation of many closed vortex loops. The intermediate phase is one in which vortex lines are unentangled, remaining on average straight and aligned with $B$, however they are in a liquid state as far as their correlations in planes perpendicular to $B$ are concerned.

As our simulations are carried out in terms of the variables $\theta_i$, we first must locate the positions of the vortex lines as they thread their way through the system. We use the following algorithm. For each plaquette $\alpha$ in the system, we compute

$$2\pi f_\alpha + \sum_\alpha [\theta_i - \theta_j - A_{ij}] \equiv 2\pi n_\alpha,$$

where the sum is taken counterclockwise around the bounds of the plaquette, $f_\alpha$ is the magnetic flux through the plaquette as given by Eqs. (12) and (13), and $[\theta_i - \theta_j - A_{ij}]$ are measured restricted to the interval $[-\pi, \pi]$. $n_\alpha$ is then the integer vorticity in $\theta_i$ going around this plaquette. As vorticity is conserved, each unit cell of the cubic mesh which has one plaquette with a vortex entering the cell has another plaquette with a vortex leaving. By computing the vorticity entering and leaving the unit cells of the mesh, we are able to trace out the paths of the vortex lines. In the event that two vortex lines enter and leave the same unit cell, we randomly assign which exiting segment is connected to which entering segment, for the purpose of identifying the paths of those lines.

1. Upper transition $T_{ce}$

One property that has received much theoretical attention has been the entanglement of vortex lines, once the ground state vortex-line lattice has melted. Interesting dynamical effects have been suggested assuming a large energy barrier exists against vortex lines cutting through each other. In such a case, entanglement would provide a topological constraint limiting vortex-line motion. A single pinned line would pin the many lines with which it is entangled, substantially reducing the flux flow resistance.
In our model, it is simple to check for entanglement. We follow a prescription proposed by Nelson.\textsuperscript{1,2} Let \{\(r_{Lz}(0), r_{Lz}(Lz), \ldots\)\} be the positions of the vortex lines in the xy plane, as they pierce the plane at \(z = 0\). Let \{\(r_{Lz}(Lz), r_{Lz}(Lz), \ldots\)\} be the corresponding locations of these vortex lines as they pierce the \(z = Lz\) plane. Since our simulation uses periodic boundary conditions in the z direction, parallel to B, all the \(r_{Lz}(Lz)\) must connect onto some \(r_{Lz}(0)\), so that the vortex lines remain continuous in the periodically extended system. If \(r_{Lz}(Lz)\) connects to \(r_{Lz}(0)\), we say that line i reconnects to itself. If we view the periodic boundary condition along the \(\hat{z}\) direction as representing the circumference of a three-dimensional torus, then line i winds only once around the torus, before reconnecting to itself. In contrast, if \(r_{Lz}(Lz)\) connects to \(r_{Lz}(0), j \neq i,\) then lines i and j are part of a larger multiple connection of lines that winds more than once around the torus, before reconnecting to itself (see Fig. 3). For an entangled vortex-line state, we would expect there to be many such multiple connections, while for disentangled approximately straight lines, there should be only simple self-reconnections. In Fig. 2, we plot the fraction of lines which reconnect to themselves, \(R\) versus temperature, which we compute while cooling the system from an initial random configuration. \(R\) drops sharply to zero at \(T_{cz}\), from its low-temperature value of one. We thus identify this upper transition, where phase coherence in the direction parallel to B is lost, as a transition from straight disentangled lines, to wandering entangled lines. Further, the fact that the multiply connected entangled lines present at high temperature completely disentangle to simple self-reconnected lines below \(T_{cz}\), suggests that the lines may easily cut through each other as they disentangle.

As an alternative criterion for vortex-line entanglement, we can consider the transverse fluctuations of the vortex lines \(\langle u^2(Lz) \rangle\) as a function of length \(0 \leq Lz \leq Lz\) parallel to B,

\[
\langle u^2(Lz) \rangle = \frac{1}{N_{t} Lz} \sum_{i=1}^{N_{t}} \langle |r_{Lz}(z) - r_{Lz}(z_0)|^2 \rangle, \quad z_0 = Lz/2,
\]

where \(r_{Lz}(z)\) is the position of line i in the xy plane at \(z\), \(N_{t} = f Lz^2\) is the number of magnetic field induced lines, and the sum is along the lengths of these lines for a distance \(Lz\). Vortex lines in our model may in general consist of two types: the magnetic field induced lines which thread the entire system from \(z = 0\) to \(z = Lz\), and thermally excited closed vortex loops. In the evaluation of Eq. (27) we do not include sums over vortex-line segments which belong to the thermal loops. In the event that a loop intersects a field induced line, we randomly assign which line segment "belongs" to the field induced line, and which segment "belongs" to the thermally excited loop.

For straight, unentangled vortex lines, we expect that \(\langle u^2 \rangle\) will saturate to a constant as \(Lz\) gets large. For wandering, entangled vortex lines we expect\textsuperscript{1} that \(\langle u^2 \rangle\) will grow with increasing \(Lz\) as

\[
\langle u^2(Lz) \rangle = u^2(T) + D(T)Lz.
\]

\(D(T)\) measures the diffusion of the line in the transverse direction, as one moves down the \(\hat{z}\) axis, and it can be used to determine the "entanglement correlation length."\textsuperscript{1}

\[
\xi_z(T) \equiv \frac{a_z^2}{4D(T)}.
\]

\(\xi_z\) measures the distance one has to move in the \(\hat{z}\) direction in order to have a transverse fluctuation equal to half the average spacing between the vortex lines. \(\xi_z\) is the length scale on which the fluctuating lines can approach each other, leading to a mutual winding and hence entangling.

In Figs. 4(a) and 4(b) we plot \(\langle u^2(Lz) \rangle\) versus \(Lz\) for several different values of temperature. For \(T \geq 2.5 J_0 \approx T_{cz}\) [Fig. 4(a)], the data points for the three largest values of \(Lz\) show a clear linear increase with \(Lz\), as expected from Eq. (28) for an entangled phase. For \(T < 2.5 J_0\) [Fig. 4(b)], there remains a systematic curvature in the data, suggesting a saturation of \(\langle u^2 \rangle\) at large \(Lz\), as in an unentangled phase; the data here is limited by finite size effects, and runs in a system with larger \(Lz\) would be

\[
\text{FIG. 3. Schematic showing how vortex lines must connect onto each other, due to the periodic boundary conditions along the } \hat{z} \text{ axis. Solid lines represent vortex lines. Dashed lines indicate how the ends of the vortex lines must match up on the } z = 0 \text{ and } z = Lz \text{ planes.}
\]

\[
\text{FIG. 4. Transverse fluctuation of vortex lines } \langle u^2 \rangle \text{ as a function of length along the line } Lz, \text{ for several different temperatures. Linear growth with } Lz \text{ indicates wandering, entangled lines. Saturation with increasing } Lz \text{ indicates unentangled lines.}
\]
needed in order to clarify the situation. Obtaining $D(T)$ from a fit of Eq. (28) to the three largest values of $l^z$, for the data of Fig. 4, we compute the resulting $\xi_z$ from Eq. (29) and show the result in Fig. 5.

In a similar manner, we can compute the “cutting correlation length,” $\xi_c$, which we define as the average distance one has to move in the $z$ direction between two successive cuttings of a given vortex line. Within our model, we define a cut as whenever there are two or more vortex lines entering and leaving a given unit cell of the numerical mesh. Such a situation represents an overlapping of the vortex cores, and there is no constraint whatsoever on how the lines will subsequently fluctuate out of this unit cell; lines are free to cut through each other, or even to cut and reconnect different segments of different lines. If $N_c$ is the average number of total cuttings observed in the simulation, then

$$\xi_c(T) \equiv \frac{N_c L_z}{N_c(T)}.$$  

(30)

We plot $\xi_c(T)$ in Fig. 5.

Comparing the entanglement and cutting correlation lengths, $\xi_z$ and $\xi_c$, we see that for $T \geq 2.5J_0 \approx T_{cz}, \xi_z \approx \xi_c$. Thus, entanglement and cutting are occurring out of the same length scales for the region above $T_{cz}$, and here we would expect entanglement to provide little constraint on the motion of vortex lines. Below $T_{cz}$, we see $\xi_c$ growing faster than $\xi_z$. However, as discussed above in connection with Fig. 4(b), we believe that our data here is limited by finite size effects, and hence the values shown in Fig. 5 are not a reliable indication of behavior in an infinite sample. It therefore remains unclear whether there exists some narrow range of temperature, at or below $T_{cz}$, in which entanglement might lead to constraints on vortex-line motion.

Finally, as an additional measure of the fluctuations of the vortex lines, we calculate the average density of vortex lines cutting the planes perpendicular to $B$,

$$n_v \equiv \frac{1}{N} \sum_i (n_i^z(i)),$$  

(31)

where the sum is over all dual sites $i$, and $n_i^z(i)$ is the vorticity passing through the plaquette in the $xy$ plane above site $i$. In virtually all cases, $n_z = 0, \pm 1$, and so $n_v$ counts the density of lines, irrespective of the sign of the vorticity. If the only vortex lines present in the system are the magnetic field induced lines, and these lines wander through the system in a directed fashion (i.e., all lines pierce each $z$ plane only once; there is no “snaking back”) then one has $n_v = f$. If, however, there are additional closed vortex-line loops excited, or lines are not directed, then $n_v$ will be larger than $f$. In Fig. 6 we plot $n_v(T)$. At low temperatures $T \lesssim 2.0J_0$, we see that $n_v \approx f$, and hence there are only directed field induced lines. However for $T \gtrsim 2.0J_0$, $n_v$ grows dramatically, becoming $n_v \approx 2f$ at the transition $T_{cz}$. Thus, by the time the transition is reached, there are many thermally excited closed vortex loops in the system. This suggests that the transition at $T_{cz}$ is mediated, not by the wandering of the field induced vortex lines, but rather by the proliferation of thermally excited vortex-line loops. The behavior of transverse fluctuations and cuttings, shown in Figs. 4 and 5, may be understood as arising from the growth of loop excitations that intersect and link up neighboring field induced lines, rather than the direct wandering of the field induced lines. In fact, Fig. 6 shows a qualitative resemblance to the density of vortices as one crosses the transition, in the ordinary three-dimensional $XY$ model\textsuperscript{30} (corresponding to $B=0$). Similarly, a plot of the specific heat $C$, which we compute by numerical differentiation of the average energy and show in Fig. 7, shows the same cusplike feature at $T_{cz}$ as in the 3D $XY$ model.\textsuperscript{30} We note that these loop excitations, which we find important for the transition at $T_{cz}$, are explicitly excluded in the “2D boson” and effective elastic medium approximations for vortex-line fluctuations, as well as in other earlier simulations.\textsuperscript{31}

2. Lower transition $T_{c\perp}$

The low-temperature phase $T < T_{c\perp}$ is characterized by an ordered vortex-line lattice, with periodicity as given in Fig. 1. The vanishing of $T_{\perp}$ above $T_{c\perp}$, to-

![FIG. 5. Correlation lengths along the direction of $B$. $\xi_z$ is the “entanglement” length, determined by the transverse fluctuations of Fig. 4. $\xi_c$ is the “cutting” length, giving the average distance between two cuttings (intersections) along a single vortex line. $\xi_c$ is the decay correlation length as extracted from the vortex-line structure function, $S(k_{\perp}, z)$ (see Sec. III C).](image)

![FIG. 6. Average density of vortex lines $n_v$, piercing planes of constant $z$. At low $T$, $n_v$ is the magnetic field induced density of $f = \frac{1}{2}$. $n_v$ increases with $T$ due to the excitation of closed vortex line loops.](image)
gether with the result that \( R \simeq 1 \) for \( T_{c_{\perp}} < T < T_{c_{\parallel}} \) (see Fig. 2), indicates that that at \( T_{c_{\perp}} \) the vortex-line lattice melts into a liquid of unentangled vortex lines. One way to measure this transition is by considering the local orientational order of vortices in the planes perpendicular to \( B \). These vortices are just the intersections of the vortex lines, with a particular plane at constant \( z \). We define the \( n \)-fold local orientational order, in terms of the quantity

\[
\Psi_n = \frac{1}{fN} \sum_{z,i} \frac{1}{M_i} \sum_{j=1}^{M_i} e^{in\phi_{j,j+1}(i)}.
\]

(32)

In the above, \( z \) labels the planes perpendicular to \( B \), \( i \) labels the vortices in the plane \( z \), and \( j \) labels the \( M_i \) nearest-neighbor vortices of \( i \) in plane \( z \). \( \phi_{j,j+1}(i) \) is the angle between the line from \( i \) to \( j \), and the line from \( i \) to \( j+1 \), which is the next neighbor after \( j \) which one encounters in rotating around \( i \) (see Fig. 8). The nearest neighbors of \( i \) are determined by the method of Voronoi polygons.\(^{32}\) For perfect square orientational order, with all \( \phi_{j,j+1}(i) = \pi/2 \), such as in the ground state of Fig. 1, we have \( \Psi_4 = 1 \), while \( \Psi_6 = e^{i\pi} \). For perfect hexatic orientational order, with all \( \phi_{j,j+1}(i) = \pi/3 \), \( \Psi_6 = 1 \), while \( \Psi_4 = e^{i4\pi/3} \). In Figs. 9(a) and 9(b), we plot, respectively, the amplitude and phase of \( \Psi_4 \) and \( \Psi_6 \). Both amplitudes show a clear kink at \( T \approx 1.3J_0 \approx T_{c_{\perp}} \). At this temperature, the phases indicate a clear crossover from the fourfold order of the ground state vortex-line lattice, to the local sixfold order of a vortex-line liquid.

For a more global picture of the transition from vortex-line lattice to vortex-line liquid, we have computed the correlation of vortices in the planes perpendicular to \( B \). If we define the vortex structure function\(^{1,9}\) as

\[
S(k_{\perp}, k_{\parallel}) \equiv \frac{1}{N} \langle n_z(k) n_z(-k) \rangle
\]

(33)

with \( k = (k_{\perp}, k_{\parallel}) \) and \( n_z(k) \) as in Eq. (17), then the correlation between vortices in different planes separated by a height \( z \), is given by the Fourier transform of Eq. (33) with respect to \( k_{\parallel} \),

\[
S(k_{\perp}, z) \equiv \frac{1}{L_z} \sum_{k_{\parallel}} S(k_{\perp}, k_{\parallel}) e^{ik_{\parallel}z}.
\]

(34)

The correlation between vortices in the same plane is then just \( S(k_{\perp}, z = 0) \). In Figs. 10(a)–10(d) we show intensity plots of \( S(k_{\perp}, z = 0) \) for \( k_{\parallel} \) in the interval \([-\pi, \pi]\), for several values of temperature, upon heating the system. At \( T = 0.6J_0 < T_{c_{\perp}} \) [Fig. 10(a)] we see sharp Bragg peaks, corresponding to the periodic vortex-line lattice of Fig. 1, superimposed on a small continuous background. The width of these Bragg peaks corresponds to the finite resolution of wave vectors allowed by our finite system, \( \Delta k_{\parallel} = 2\pi/a_{\parallel} \). At \( T = 1.5 > T_{c_{\perp}} \) [Fig. 10(b)], we see an approximately rotationally invariant structure, with peaks of finite width. The first peak is at a radius \( k_{\parallel} = 2\pi/a_{\parallel} \), corresponding to the average separation between vortices; the smaller, second peak, is at \( k_{\parallel} = 4\pi/a_{\parallel} \). Such a structure function is characteristic of a liquid. The lack of complete rotational symmetry in the peaks (aside from a slight fourfold distortion to be expected from the underlying cubic mesh) we believe is due to the finite time duration of our simulation; we see no evidence for long range orientational order as might be present in a hexatic line liquid.\(^{33}\) As \( T \) increases [Fig. 10(c)], the second peak disappears into the continuous background. At \( T = 2.5J_0 \approx T_{c_{\parallel}} \) [Fig. 10(d)], both peaks have virtually disappeared, and only a continuous background remains. Thus for \( T > T_{c_{\parallel}} \), even the short range order, characterized by the average spacing between field induced vortex lines, is lost; this is consistent with the proliferation of many vortex-line loops, as evidenced by Fig. 6.

In the intermediate phase, we find that the system re-
FIG. 10. Intensity plots of vortex structure function $S(k_x, z = 0)$, for various temperatures. $k_x$ is the horizontal, and $k_y$ is the vertical direction. Brightness measures the magnitude of $S$, as in a diffraction pattern. The $\delta$-function peak at $k_x = 0$ is the temperature independent constant $S(0, z = 0) = f^b L^2$, which sets the relative intensity scale. (a) shows a lattice of Bragg peaks, characteristic of a vortex-line lattice, at $T < T_{c\perp}$. (b) and (c) are characteristic of a vortex-line liquid, for $T_{c\perp} < T < T_{c\perp}$. (d) is for $T > T_{c\perp}$, and the dominating continuous background indicates the proliferation of closed vortex-line loop excitations.

laxes very slowly to equilibrium. We illustrate this by computing the time dependent correlation

$$g(t) = \frac{1}{N} \sum_i \{ \langle n_z(i, t)n_x(i, 0) \rangle - \langle n_z(i, t) \rangle \langle n_x(i, 0) \rangle \},$$

(35)

where the time $t$ is just the number of Monte Carlo sweeps through the entire cubic mesh. We plot $g(t)$ at $T = 1.9J_0$ in Fig. 11. After an initial fast decay, we observe a slow algebraic decay, $g(t) \sim t^{-\zeta}$ with $\zeta \approx 0.23$. We believe that it is this very slow decay which is responsible for the small but finite values of $T_{c\perp}$ that are seen for $T > T_{c\perp}$ in Fig. 2, rather than usual finite size effects. When we doubled the number of Monte Carlo steps in our simulation, we observed $T_{c\perp}$ in this region to decrease.

Because of this slow equilibration, as the system is cooled back below $T_{c\perp}$, we have been unable to regain the ordered vortex-line lattice of Fig. 1. Instead, we cool into a structure consisting of perfectly straight lines, ordered into local domains of the ground state, with frozen in domain walls. In Fig. 12 we show an intensity plot of $S(k_x, z = 0)$ at $T = 0.6J_0$ for such a cooled configuration. The two rings of peak intensity which exist for $T > T_{c\perp}$ [see Fig. 10(b)], are now narrow and “squared off.” This feature of freezing in domain walls, as well as the hysteresis in $T_{c\perp}$ seen in Fig. 2, suggest that the tran-
FIG. 11. Time decay of real space vortex correlation function of Eq. (35) for $T = 1.9 J_0$, in the intermediate phase. The linear behavior on the log-log plot indicates algebraic decay with the power 0.23. The unit of time is one Monte Carlo sweep through the entire numerical mesh. The solid line is a guide to the eye.

sition at $T_{c1}$ may be first order. Another possibility is that $T_{c1}$ is not a true thermodynamic transition at all, but only a cross-over depinning temperature where the effective energy barriers to vortex-line motion grow so large that vortex motion is frozen out on the scales of our simulation.

C. Vortex structure function

In Fig. 10, we used the structure function $S(k_\perp, k_z)$ to qualitatively determine the nature of the vortex-line structure in the different thermodynamic phases. Here we present a more detailed analysis of $S(k_\perp, k_z)$, from which we can extract information about the effective elastic constants of the model.

Following the works of Marchetti and Nelson and LeDoussal, the natural form for the coarse-grain averaged free energy of our model, valid on hydrodynamic length scales $k_\perp^{-1} > a_0$, in the vortex-line liquid phase, is

$$F = \frac{1}{2Nf^2} \sum_k \left( c_{44}(k) \sum_{\mu, \nu = x, y} [\delta n_{\mu, \nu}(k) \delta n_{\mu, \nu}(-k)] + c_1(k) \delta n_x(k) \delta n_z(k) \right),$$  \hspace{1cm} (36)

where $\delta n_\mu(k)$ give the deviation from the average uniform density $f_\mu$ and are now viewed as independent continuous variables. $c_{44}$ and $c_1$ are the tilt and bulk moduli of the vortex lines on the numerical mesh.

Using the constraint that vortex lines must be continuous, i.e., the discrete divergence of $n_\mu(i)$ vanishes,

$$\sum_{\mu} [n_\mu(i + \hat{\mu}) - n_\mu(i)] = 0$$ \hspace{1cm} (37)

one can do the appropriate Gaussian integrals over $\delta n_\mu(k)$ and compute the structure function of Eq. (33):

$$S(k_\perp, k_z) = \frac{T f^2 K^2}{c_1(k) K^2 + c_{44}(k) K_z^2},$$ \hspace{1cm} (38)

where $K_\mu$ is related to $k_\mu$ as in Eq. (16). The factor $K^2_\perp$ in the numerator of the expression above determines the qualitative form of the continuous background of $S(k_\perp, k_z = 0)$ seen in Figs. 10.

From Eq. (36) and the form of the bare vortex-line interaction given in Eqs. (16) and (18), we expect

$$c_1(k) = 4\pi^2 f^2 J_1 G_k, \quad c_{44}(k) = c_1(k) + f \epsilon_1(k_z),$$ \hspace{1cm} (39)

where we write $J_1$ instead of $J_0$ to allow for possible renormalization of this coupling in the coarse-graining procedure. $\epsilon_1$ is the single vortex-line tension, which depends on $k_z$, due to the interaction between the different segments of the single line. The difference $c_{44} - c_1$ just gives the additional energy needed to create the elongation of the vortex lines described by the transverse components of the fluctuation. Substituting Eq. (39) into Eq. (38) gives

$$TK^2_\perp / S(k_\perp, k_z) = 4\pi^2 J_1 + [\epsilon_1(k_z)] f K^2_z.$$ \hspace{1cm} (40)

We now compute $S(k_\perp, k_z)$ directly within the Monte Carlo simulations. Calculations are done for values of $k_z = 2\pi m_z/L_z$, $m_z = 0, 1, ..., L_z/2$, and for the special values of $k_\perp = k_\perp \hat{x}$ and $k_\perp \hat{y}$, where $k_\perp = 2\pi m_\perp / L_\perp$, $m_\perp = 1, ..., L_\perp / 2$. By symmetry, we expect that $S(k_\perp \hat{x}, k_z) = S(k_\perp \hat{y}, k_z)$, hence we combine our results by defining $S(k_\perp, k_z)$ as the average of $S$ in these two special directions.

In Figs. 13(a)–13(f), we plot $TK^2_\perp / S$ as a function of $K^2_\perp$ and $K^2_z$ for temperatures $T = 1.0 J_0$ (in the vortex-line lattice phase), $T = 2.25 J_0$ (in the intermediate phase), and $T = 2.75 J_0$ (in the normal phase). In the line lattice phase [Figs. 13(a) and 13(b)], we see that

FIG. 12. Intensity plot of vortex structure function $S(k_\perp, z = 0)$, for $T = 0.6 J_0$, upon cooling $T_{c1}$.
TK^2 / S is approximately independent of \( k_\perp \), obeying the hydrodynamic form of Eq. (40) at all \( k \). In the intermediate, unentangled line liquid phase [Figs. 13(c) and 13(d)], a strong \( k_\perp \) dependence appears, with a minimum appearing at \( K_1^2 \approx 1.2 \), very close to the value of 1.38 which corresponds to \( k_\perp = 2\pi / \alpha_0 \). In the normal phase [Figs. 13(e) and 13(f)] as \( T \) increases, this minimum decreases to smaller \( k_\perp \), reflecting the increasing density of vortex lines [see Fig. 6; this also explains why the minimum in Fig. 13(c) is not exactly given by \( k_\perp = 2\pi / \alpha_0 \)].

This minimum as a function of \( k_\perp \) is a reflection of the peak in \( S(k_\perp, k_z) \) which occurs isotropically in the \( k_\perp \) plane at \( |k_\perp| \approx 2\pi / \alpha_0 \), in the line liquid phase. It may be viewed as an effective softening of the bulk modulus \( c_{44}(k) \) at these large values of \( k_\perp \). In the line lattice phase, this isotropic peak contracts into discrete Bragg peaks, none of which lie along the \( \vec{X} \) or \( \vec{Y} \) directions [see Fig. 10(a)]. Hence the \( k_\perp \) dependence of \( TK^2 / S \) flattens out.

Although Eqs. (38)–(40) were proposed for the line liquid phase, similar results have been given for the line lattice phase, using an effective elastic approximation. In this case the only change is that one must replace \( c_1 \) in Eq. (38) with \( c_{11} = c_1 + c_{66} \), where \( c_{66} \) is the shear modulus. This results in the addition of the term \( c_{66}K^2 \) to Eq. (40). It is unclear whether or not the differences observed in the \( k_\perp \) dependence of Figs. 13(a) and 13(c), may be interpreted in terms of the vanishing of the shear modulus at the melting transition \( T_c \).

Comparing the curves of \( TK^2 / S \) versus \( K_1^2 \), for different \( k_\perp \), in Figs. 13(d) and 13(f), one sees that the dependence of \( TK^2 / S \) on \( k_\perp \) and \( k_x \) cannot be separated out into the simple form \( 4\pi J_1(k_\perp) + (\epsilon_1(k_x)/f)K_2^2 \) as might be expected from the “2D boson” approximation. As we have noted in earlier work, the more complex dependence we observe may be interpreted as adding a \( k_\perp \) dependence to the line tension \( \epsilon_1 \). Using Eq. (40) to define the effective line tension as \( k_\perp \rightarrow 0 \),

\[
\epsilon_1(k_x) \equiv \frac{f}{K_2^2} \left[ \frac{TK^2}{S(0, k_x)} - \frac{TK^2}{S(0, 0)} \right],
\]

we plot in Fig. 14 \( \epsilon_1(k_x) \) versus \( \ln K_2^2 \) for several temperatures \( T \). The plots approach straight lines at large \( K_2^2 \). This is in qualitative agreement with the expression for the line tension obtained from an elastic approximation, expanding about the line lattice in a layered superconductor,

\[
\epsilon_1(k_x) = \frac{\sigma_0}{2(4\pi\lambda)^2} \ln \left[ \frac{k_{l,\text{max}}^2}{k_{l,\text{min}}^2 + K_2^2} \right],
\]

where for the line lattice, \( k_{l,\text{max}} \approx \pi / \xi_0 \), and for \( \lambda \gg \alpha_0 \),

\( k_{l,\text{min}} \approx \pi / \alpha_0 \). For our data, we find a reasonable fit with \( k_{l,\text{max}} \approx \pi \) (in our model \( \xi_0 = 1 \)), but \( k_{l,\text{min}} \approx 0 \).

From the above analysis, we can extract several parameters of interest. Considering the \( (k_\perp, k_x) \rightarrow 0 \) limits, we fit our data to the form of Eq. (40) and extract the values of the hydrodynamic parameters \( J_1 \) and \( \epsilon_1(0) \), which we plot versus \( T \) in Figs. 15(a) and 15(b), respectively. Defining the prefactor of the logarithm in Eq. (42) as...
FIG. 14. Line tension $\varepsilon_1(k_z)$, from Eq. (41), using data as in Fig. 13. The linear dependence on $\ln K_2^2$, at large $K_2^2$, agrees qualitatively with the theoretical prediction of Eq. (42). $K_2^2$ is related to $k_z$ as in Eq. (16).

\[ (\pi/2) J_2 \] 

as motivated by Eq. (8), we can fit the large $K_2^2$ part of the data in Fig. 14 to $\sim \ln(K_2^2)$ and plot the resulting $J_2$ in Fig. 15(a). We see that $J_1$ and $J_2$ are of comparable magnitude, except at the very lowest temperature $T = 0.6 J_0$, and above $T_{ce} \approx 2.5 J_0$, where $J_2$ decays to zero, while $J_1$ remains approximately constant.

Finally, we can extract a correlation length along the field direction by computing the Fourier transform of $S(k_{\perp}, k_z)$ with respect to $k_z$, as in Eq. (34). Fitting the resulting $S(k_{\perp}, z)$, at large $z$, to the form

\[ S(k_{\perp}, z) \approx S_0(k_{\perp}) \left[ e^{-z/\xi(k_{\perp})} + e^{-(L_z-z)/\xi(k_{\perp})} \right] \tag{43} \]

(this form being chosen so as to account for the periodic boundary conditions along $x$), we plot the correlation length $\xi_z \equiv \xi(k_{\perp} = 2\pi/a_x)$ versus $T$ in Fig. 5. For $T < 2.25 J_0$ we found no measurable decay in $S(k_{\perp}, z)$. For $T \geq 2.25 J_0$ we find $\xi_z \approx \xi_z$, the cutting correlation length. This reinforces our earlier observation that cutting and entangling appear to occur on the same length scales. We can use the value of $\xi_z$ just obtained to estimate the hydrodynamic line tension. For finite $\xi_z$, we expect that the effective $k_z \rightarrow 0$ line tension can be obtained from Eq. (42) (appropiate for the line lattice where $\xi_z \rightarrow \infty$) by imposing a small $k_z$ cutoff at $k_z \approx \pi/\xi_z$. Hence we estimate

\[ \varepsilon_1(0) \approx \varepsilon_1(k_z \approx \pi/\xi_z) \approx \pi J_2 \ln \xi_z. \tag{44} \]

We plot the result in Fig. 15(b), where we have set $\xi_z \equiv L_z$ for all $T < T_{ce}$. Although we observe quite reasonable agreement with the hydrodynamic result obtained from a direct fit to Eq. (40), we note that the dominant temperature dependence comes from the behavior of $J_2$. We do not have large enough of a range of $\xi_z$ to provide a sensitive test of the $\ln \xi_z$ dependence.

IV. CONCLUSIONS AND DISCUSSION

To conclude, we have studied a model of fluctuating vortex lines, and find three distinct thermodynamic phases. Upon heating, the system has sharp transitions from a vortex-line lattice, to an unentangled vortex-line liquid, to an entangled vortex-line liquid with many closed vortex-line loops and much vortex-line cutting. The lower transition is associated with the loss of phase coherence in the superconducting order parameter, in directions perpendicular to the applied magnetic field. The upper transition is associated with the loss of phase coherence in the parallel direction. This behavior is consistent with that argued by Feigel’man and co-workers. Similar behavior has been argued by Glazman and Koshelev, for low magnetic fields $B < B_c$. Nelson originally suggested an unentangled line liquid phase as a finite size effect, when $\xi_z \approx L_z$. We cannot rule out the possibility that the difference we observe between $T_{ce}$ and $T_{ce}$ is such a finite size effect. However, if this is so, our observation that $T_{ce} \approx 2T_{ce}$ would indicate a rather large critical region about the single transition of the infinite system. Similarly, more detailed finite size studies would be needed for one to clarify the nature of the phase transitions in our model. In our earlier work, at the higher density $f = \frac{1}{3}$, we found evidence for only a single transition. It may be that there is indeed only a single transition at higher densities, or it may be that the width of the intermediate region becomes too narrow to distinguish. It is interesting to note that in a related model, the 3D $XY$ model of stacked triangular planes with antiferromagnetic coupling within the triangular planes and ferromagnetic coupling between the planes, which has the same symmetry as our model with $f = \frac{1}{3}$, there is only a single transition, however, it is a tetracritical point.

Our model has involved two key approximations in its mapping to a realistic superconductor. One is the dis-
cretization of space into a cubic mesh, on the length scale of the bare coherence length $\xi_0$. The discretization along the $\hat{z}$ axis may be viewed as representing the layered structure of the high-$T_c$ materials, as in a Lawrence-Doniach model.\textsuperscript{39} The discretization in the $xy$ plane, however, represents an artificial periodic pinning potential for vortex-line fluctuations. Our hope was that this pinning potential primarily acts to remove the modes associated with a uniform translation of the entire vortex-line lattice. However the “forces” associated with this pinning potential are strong enough to tilt the balance in favor of a square line lattice in the ground state, as opposed to the triangular lattice expected in a continuum. One may worry that these forces also significantly suppress vortex-line fluctuations, leading to a stabilization of the line lattice, as well as phase coherence of the superconducting order parameter, at low temperatures. However, in Figs. 15 we see a sharp drop in $J_1$ and $\epsilon_l(0)$ at $T < 1.0 J_0$, noticeably below the lower transition $T_{c1} \approx 1.3 J_0$. If we interpret $T$ as the energy scale associated with the energy barriers of the periodic pinning potential, then one expects that the transition at $T_{c1}$ is a true melting transition of the line lattice, rather than just a depinning of the lines.

The second approximation has been the assumption of a uniform magnetic induction $B$ inside the superconductor, i.e., the $\lambda \to \infty$ limit. We first discuss the effects of this approximation on vortex-line fluctuations. As mentioned in Sec. II, setting $\lambda \to \infty$ suppresses long wavelength (small $k < \lambda^{-1}$) density fluctuations, while being a good approximation for high $k \approx \lambda^{-1}$ fluctuations. It is generally believed,\textsuperscript{4,5} however, that it is high $k \perp 2\pi/a_v$ shear fluctuations, which do not alter the long wavelength density, that are responsible for melting of the vortex-line lattice. Ikeda \textit{et al.}\textsuperscript{40} specifically note that the melting transition, as determined from a Lindemann criterion, is largely unaffected when comparing the finite $\lambda$ to the $\lambda \to \infty$ cases, providing $\lambda \gg a_v$. Hence we expect behavior in the line lattice case, and not melting at $T_{c1}$, to be largely independent of this uniform $B$ approximation. Similarly, in the normal, entangled line liquid phase, $T > T_{c2}$, one expects this approximation to be good for describing local properties (such as entangling, cutting, and $S(k)$ for $k > \lambda^{-1}$) provided $\lambda \gg \xi_z$, the correlation length along the $\hat{z}$ axis. In this case, fluctuations impose a shorter cutoff on effective interactions than does magnetic screening. It is less clear how valid the approximation is in the intermediate phase, where $\xi_z \to \infty$, and there is no other natural length scale along the $\hat{z}$ direction with which to compare $\lambda$. However we note that Peigel\textsuperscript{35} predicts a similar unentangled phase, working within the “2D boson” approximation, in which the interaction between vortex lines along the $\hat{z}$ direction is short ranged. Further work will be necessary to determine whether the unentangled line liquid phase observed here, is stable against fluctuations of $B$.

The effect of the $\lambda \to \infty$ approximation on fluctuations of the phase of the superconducting order parameter, and the existence of true long range order in the gauge invariant order parameter correlation function, has been a subject of much discussion.\textsuperscript{5,40-42} Moore\textsuperscript{41} and Ikeda \textit{et al.}\textsuperscript{40} have shown that in three dimensions, the gauge invariant order parameter correlations decay algebraically, in the line lattice phase, for the $\lambda \to \infty$ case. However for the finite $\lambda$ case, they decay exponentially. Such algebraic decay is consistent with a finite helicity modulus\textsuperscript{43} $Y$, as we observe in our model. Exponential decay, for finite $\lambda$, would imply $Y \to 0$. However as shown in Refs. 40 and 41, the length scale for such exponential decay is given by $(\pi J_0/T)\lambda \xi_z$, and is hence comparable to the size of macroscopic samples, for the large $\kappa$ high-$T_c$ materials. Thus the $\lambda \to \infty$ limit should be appropriate in such situations.\textsuperscript{44}

Although we have not attempted any direct mapping between our model and a specific high-$T_c$ material (we took isotropic couplings, and ignored all $T$ dependences of our bare parameter $J_0$), it is tempting to try to discuss resistivity measurements in very pure YBCO single crystals,\textsuperscript{7,65} in terms of the behavior we observe in our model. Above the upper transition $T_{c2}$, where there are many vortex-line loops, much wandering and entangling of the field induced vortex lines, and easy cutting of vortex lines, we might expect substantial flux flow resistance due to the flow of effectively independent vortex-line segments. This region possibly describes much of the broadening of the resistive transition, in comparison to the $B = 0$ case, and could be explained with traditional flux flow models.\textsuperscript{46} Resistivity, $\rho_\perp$, for current $I \parallel B$, should be comparable to $\rho_\perp$ for $I \perp B$ (apart from effects due to explicit anisotropy of couplings), as there are almost as many vortex-line segments oriented perpendicular to $B$ as parallel to $B$ (from closed loops, and transverse segments of wandering lines). As $T$ is cooled towards $T_{c2}$ we would expect $\rho_\perp$ to decrease substantially compared to $\rho_\parallel$, as the vortex lines disentangle, and straighten parallel to $B$, and there are fewer loop excitations. Such behavior is consistent with that seen in experiments\textsuperscript{45} where $B$ is applied in the $ab$ (CuO) planes, and hence there is no explicit anisotropy effects for $\rho_\perp$ versus $\rho_\parallel$. Below $T_{c2}$, the linear resistivity $\rho_\perp$ should vanish, however the linear part of $\rho_\parallel$ would remain finite. The transition at $T_{c2}$ might offer an explanation for the “kink” observed in $\rho_\parallel$ in the tail of the resistance curve, for very pure samples with $B$ along the $c$ axis.\textsuperscript{7} Finally, $\rho_\perp$ would vanish below $T_{c1}$.

ACKNOWLEDGMENT

This work was supported by Grant No. DE-FG02-89ER14017 from the U.S. Department of Energy.

17. A mapping that allows one to obtain results for systems with anisotropic couplings, in terms of an equivalent isotropic system, is given in G. Blatter, V. B. Geshkenbein, and A. I. Larkin, Phys. Rev. Lett. 68, 875 (1992).
33. H. S. M. Coxeter, Introduction to Geometry (Wiley, New York, 1961), p. 53. In order to ensure that we do not triangulate square unit cells, we use the following prescription: Each pair of points $i$ and $j$ are joined with a tentative bond. For every other point $k$ within a fixed range of $i$ and $j$, we measure the angle $i k j$. If this angle is larger or equal to 90°, we remove the bond $i j$.
35. For a finite-size analysis that indicates this transition is first order in a closely related model, see E. H. Hettzel, A. Sudbø, and D. A. Huse, Phys. Rev. Lett. 69, 518 (1992).
37. However, according to their estimate of $B_{cr}$ [see Eq. (5) of Ref. 5], we believe our model lies in the region $B > B_{cr}$.
43. Such is the case, for example, in the Kosterlitz-Thouless transition of the ordinary 2D XY model; see T. Ohta and D. Jasnow, Phys. Rev. B 20, 139 (1979).
44. In Ref. 40 it is argued that for both the finite $\lambda$ and the $\lambda \rightarrow \infty$ cases, the order parameter correlations, both perpendicular and parallel to $B$, become short ranged once the shear modulus $\varepsilon_{06} \rightarrow 0$. This would appear to contradict the observation in our model that $T_{c} > 0$ in the intermediate phase, where presumably $\varepsilon_{06} = 0$. Our results also contradict the prediction of Ref. 5, that phase coherence parallel to $B$ is lost at a temperature much below the melting transition of the vortex-line lattice.
FIG. 10. Intensity plots of vortex structure function $S(k, z = 0)$, for various temperatures. $k_x$ is the horizontal, and $k_y$ is the vertical direction. Brightness measures the magnitude of $S$, as in a diffraction pattern. The $\delta$-function peak at $k = 0$ is the temperature independent constant $S(0, z = 0) = J^2 L^2$, which sets the relative intensity scale. (a) shows a lattice of Bragg peaks, characteristic of a vortex-line lattice, at $T < T_{v\perp}$. (b) and (c) are characteristic of a vortex-line liquid, for $T_{v\perp} < T < T_{v\perp}$. (d) is for $T > T_{v\perp}$, and the dominating continuous background indicates the proliferation of closed vortex-line loop excitations.