Lagrangian Equations with Constraints

In previous examples we used constraints to eliminate degrees of freedom, and this to write the Lagrangian in terms of only independent degrees of freedom.

Here we will keep all the degrees of freedom and introduce the constraints by the method of Lagrange multipliers. We will see that the Lagrange multiplier is related to the forces that impose the constraints.

We saw for Euler's eqn with constraints:

\[ \frac{\partial L}{\partial \dot{q}_c} - \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}_c} \right) + \sum_{k=1}^{m} \lambda_k(t) \frac{\partial g_k}{\partial q_c} = 0 \quad \varepsilon = 1, \ldots, N \]

\[ \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_c} \right) - \frac{\partial L}{\partial q_c} = \sum_{k=1}^{m} \lambda_k(t) \frac{\partial g_k}{\partial q_c} \]

\( q_i \) are the generalized coordinates of the problem.

When \( g_i \) is an ordinary rectangular spatial coordinate, then \( \sum \lambda_k(t) \frac{\partial g_k}{\partial q_c} \) is a force — as we saw

in our derivation of Lagrange's eqn from Newton’s 2nd law, we can see that this is dimensionally
correct since \( L \) has units of energy, and \( g_i \) has units of length, then \( \frac{d}{dt} (g_i \cdot \hat{g}_i) = \frac{\partial L}{\partial g_i} \).

The units of energy/length = units of force.

If \( g_i \) is not a length, then \( \sum_{k=1}^{n} \lambda_k(t) \frac{\partial \hat{g}_i}{\partial \hat{g}_i} \) is called a "generalized force".

For example, if \( \theta \) is an angle, then \( \sum_{k=1}^{n} \lambda_k(t) \frac{\partial \hat{g}_i}{\partial \hat{g}_i} \) is units of torque. Generalized force for \( \hat{g}_i \) is denoted "\( Q_i \)."

\( Q \cdot \delta \hat{g}_i \) is work done by generalized force as coordinate changes by \( \delta \hat{g}_i \).

**Examples**

1. Inclined plane
   - \( g(x, y) = y = 0 \)
   - Block stays on plane by generalized force as coordinate changes.

Instead of using this constraint to eliminate \( y \) from Lagrangean \( \mathcal{L} \), we keep both \( x \) and \( y \) and use Lagrange multipliers to handle constraint.

\[ \mathcal{T} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]
\[ \mathcal{U} = -mg (\sin \theta \  \hat{x} - \cos \theta \  \hat{y}) \]

\[ \hat{\mathcal{L}} = \mathcal{T} - \mathcal{U} \]

\[ \hat{\mathcal{L}} \] follows since vertical direction \( \hat{z} \) in \( x-y \) coordinates is

\[ \hat{e} = \cos \theta \  \hat{y} - \sin \theta \  \hat{x} \]

So height \( \hat{z} \cdot \hat{e} = -x \sin \theta + y \cos \theta \).
\[ \dot{x} = r - u = \frac{1}{2} m (x^2 + y^2) + mg (x \sin \theta - y \cos \theta) \]

\[ \frac{d}{dt} \left( \frac{2x}{\partial x} \right) - \frac{2x}{\partial x} = a \frac{2x}{\partial x} \]

\[ y(x, y) = y = 0 \]

\[ \Rightarrow m \ddot{x} - mg \sin \theta = 0 \quad (1) \]

\[ \frac{d}{dt} \left( \frac{2y}{\partial y} \right) = a \frac{2y}{\partial y} \]

\[ \text{since } \frac{2y}{\partial y} = 1 \]

\[ \Rightarrow m \ddot{y} + mg \cos \theta = a \quad (2) \]

(1) \[ m \ddot{x} = mg \sin \theta \Rightarrow \ddot{x} = g \sin \theta \quad \text{accel down plane as before} \]

(2) \[ m \ddot{y} + mg \cos \theta = a \]

\[ \text{use constraint } y = 0 \Rightarrow \dot{y} = 0 \]

\[ \Rightarrow (2) \Rightarrow mg \cos \theta = a \]

Forces of constraint are - give the forces not included
\[ \text{along } x: \quad F_x = a \frac{2x}{\partial x} = 0 \]

\[ \text{along } y: \quad F_y = a \frac{2y}{\partial y} = a = mg \cos \theta \]

\[ N = mg \cos \theta \]

\[ \text{This is just the normal force, which is in the } y \text{ direction!} \]
constraint of fixed length 
\[ g(r, \phi) = r - \ell = 0 \]

\[ T = \frac{1}{2} m (r^2 + r^2 \dot{\phi}^2) \]

\[ U = -mg \rho \cos \phi \]

\[ L = T - U = \frac{1}{2} m (r^2 + r^2 \dot{\phi}^2) + mg \rho \cos \phi \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = \lambda \frac{\partial \phi}{\partial \phi} \]

\[ \frac{\partial L}{\partial \phi} = m r^2 \ddot{\phi} + 2 m r \dot{\phi} \dot{\rho} + m g r \sin \phi = 0 \]

\[ \frac{\partial L}{\partial \rho} = m r^2 \ddot{\rho} \]

\[ \Rightarrow m r^2 \ddot{\phi} + 2 m r \dot{\phi} \dot{\rho} + m g r \sin \phi = 0 \]

since \( \frac{\partial \phi}{\partial \phi} = 0 \)

\[ \frac{\partial L}{\partial \rho} = m r^2 \ddot{\rho} = \lambda \]

since \( \frac{\partial \rho}{\partial \rho} = 1 \)

constraint \( r = \ell \) \( \Rightarrow \dot{r} = 0 \), \( \ddot{r} = 0 \)

substitute into (1) \( \Rightarrow m \ell^2 \dot{\phi} + mg \ell \sin \phi = 0 \)

\( \Rightarrow \dot{\phi} + g \sin \phi = 0 \) as found before

substitute into (2) \( \Rightarrow -m \ell \dot{\phi}^2 - mg \cos \phi = \lambda \)

Generalized forces of constraint

along \( \phi \): \( Q_\phi = \lambda \frac{\partial \phi}{\partial \phi} = 0 \) = torque

along \( r \): \( Q_r = \lambda \frac{\partial \rho}{\partial r} = \lambda \) = radial force

\[ = -mg \cos \phi - m \ell \dot{\phi}^2 \]
Note: the force we find above is just what we expect.

For the circular motion, the radial component of acceleration is just the centripetal acceleration
\[ a_r = \omega^2 r. \]
So \(-m \omega^2 \phi\) is the net force in radial direction \(F_r\).
\[ F_r = mg \cos \phi - T, \]
where \(T\) is the tension in rope of length \(l\).

\[ \Rightarrow F_r = mg \cos \phi - T = -m \omega^2 \phi. \]

\[ \Rightarrow \text{force of constraint} \quad -T = -m \omega^2 \phi - mg \cos \phi = \mathcal{F}. \]

(C since force computed is always in the direction of increasing values of the coordinate, we have that the force of constraint \(\frac{\partial \mathcal{F}}{\partial r} = -T\) rather than \(T\).

Force of constraint is always that part of the total force that has not been included in the potential energy \(U\) that entered the Lagrangian \(\mathcal{L}\).

In this case, this is just the tension \(T\).
More interesting examples

3. Disk rolling without slipping down incline plane.

- $y$ gives height of center of disk from top of plane.
- $\alpha$ gives angle of rotation of disk.

To compare the methods, we can solve this problem the old Newtonian way.

Force balance along $y$ - parallel to surface:

\[ M'' = mg \sin \alpha - F_f \]

$F_f$ is friction.

Force balance perpendicular to surface:

\[ 0 = -mg \cos \alpha + N \]

$N$ is normal force.

Torque about center of disk:

\[ F_f R = I \theta \]

$I$ is moment of inertia.

No slipping constraint:

\[ y = R \theta \]

\[ \Rightarrow \ y'' = R \theta'' \]

Substitute into 3 to get:

\[ F_f R = \frac{I}{R} y'' \]

\[ \Rightarrow F_f = \frac{I}{R^2} y'' \]
Substitute \( \Theta \) to get:

\[
M \ddot{y} = M g \sin \alpha - \frac{I}{R^2} \ddot{y}
\]

\[
(M + \frac{I}{R^2}) \ddot{y} = M g \sin \alpha
\]

\[
\ddot{y} = \frac{M g \sin \alpha}{M + \frac{I}{R^2}} = \frac{g \sin \alpha}{1 + \frac{I}{MR^2}}
\]

for a solid circular disk, \( I = \frac{1}{2} MR^2 \) \( \Rightarrow \) \( \ddot{y} = \frac{2}{3} g \sin \alpha \)

Frictional force \( F = \frac{I}{R^2} \ddot{y} = M g \sin \alpha = \frac{1}{3} M g \sin \alpha \)

Now solve Lagrange's way.

\[
T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2
\]

\[
U = -M g y \sin \alpha \quad \text{(Origin is at top of incline)}
\]

Constraint \( g(y, \theta) = y - R \theta = 0 \)

\[
\ell = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2 + M g y \sin \alpha
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 2 \frac{\partial \theta}{\partial y}
\]

\[
\Rightarrow M \ddot{y} - M g \sin \alpha = 2 \quad \text{as } \frac{\partial \theta}{\partial y} = 1
\]
\[
\frac{d}{dt} \left( \frac{2x}{2\theta} \right) - \frac{2x}{2\theta} - 2 \frac{2\theta}{2\theta} = 0
\]

\[
\Rightarrow \quad I \ddot{\theta} = -AR \quad \text{as} \quad \frac{2\dot{\theta}}{2\theta} = -R
\]

(1) \quad M \ddot{y} - Mg \sin \alpha - \lambda = 0

(2) \quad I \ddot{\theta} = -AR

(3) \quad \dot{y} = R \dot{\theta}

Substitute (3) \Rightarrow \ddot{y} = R \ddot{\theta}

Substitute into (2) to get

\[
I \ddot{\theta} = -AR \quad \Rightarrow \quad \lambda = \frac{-I \ddot{\theta}}{R^2}
\]

Substitute into (1) to get

\[
M \ddot{y} - Mg \sin \alpha = \frac{-I \ddot{\theta}}{R^2}
\]

\[
\Rightarrow \quad (M + \frac{I}{R^2}) \ddot{y} = Mg \sin \alpha
\]

\[
\ddot{y} = \frac{G \sin \alpha}{1 + \frac{I}{MR^2}} \quad \text{as in Newtonian solution}
\]

So

\[
\lambda = \frac{-I \ddot{\theta}}{R^2} = \frac{-I}{R^2} \frac{G \sin \alpha}{1 + \frac{I}{MR^2}} = -\frac{G \sin \alpha}{R^2 + \frac{I}{MR^2}}
\]

\[
\Rightarrow \quad \lambda = -\frac{Mg \sin \alpha}{I + \frac{MR^2}{M}}
\]
generalized forces of constraint

in y direction: \( Q_y = F_y = \lambda \frac{2 \beta}{2y} = \lambda \frac{-M g \sin \alpha}{1 + \frac{R^2}{I}} \)

Same as in Newtonian solution
- the minus sign is because \( F_y \) points in negative y direction

in \( \theta \) direction: torque \( \tau = \frac{d}{d \theta} \int_{0}^{\theta} R \, d\theta = -R \lambda \frac{2 \beta}{2y} \)

\( \tau = -F_y R = R |F_y| \)

this is just the torque due to the frictional force
Mass sliding on frictionless hemispherical surface

\[ T = \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2) \]

\[ U = m g r \cos \theta \]

\[ L = \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2) - m g r \cos \theta \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \frac{\partial}{\partial \theta} \left( \frac{\partial L}{\partial \dot{\theta}} \right) \]

\[ \Rightarrow m \ddot{r} - m r \dot{\theta}^2 + m g \cos \theta = 2 \frac{\partial g}{\partial r} \] since \( \ddot{\theta} = 0 \)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 2 \frac{\partial g}{\partial \theta} \]

\[ \Rightarrow m r \ddot{\theta} + 2 m r \dot{r} \dot{\theta} - m g r \sin \theta = 0 \] since \( \frac{\partial g}{\partial \theta} = 0 \)

constraint \( r = R \) \( \Rightarrow \)

from 3, \( \dot{r} = 0 \), \( \dot{\theta} = 0 \)

Substitute into 1 \( \Rightarrow -m r \dot{\theta}^2 + m g \cos \theta = 0 \)

2 \( \Rightarrow m r \ddot{\theta} - m g R \sin \theta = 0 \)
\[ \dot{\theta} = \frac{g}{k} \sin \theta \]

We can integrate this using the following trick:

\[ \dot{\theta} \frac{d\theta}{dt} = \frac{d}{dt} \theta \frac{d\theta}{d\theta} = \frac{d\theta}{d\theta} \dot{\theta} \]

\[ \int \dot{\theta} d\theta = \int \frac{g}{k} \sin \theta d\theta \]

\[ \frac{1}{2} \theta^2 = \frac{g}{k} \left(1 - \cos \theta \right) \]

Constant of integration set by assuming \( \dot{\theta} = 0 \) when \( \theta = 0 \)

(Mass starts at rest on top of hemisphere)

\[ \dot{\theta}^2 = \frac{2g}{k} (1 - \cos \theta) \]

\[ \Rightarrow \ddot{\theta} = mg \cos \theta - mR \dot{\theta}^2 = mg \cos \theta - mR \frac{2g}{k} (1 - \cos \theta) \]

\[ = mg \left( \cos \theta - 2 + 2 \cos \theta \right) = mg \left(3 \cos \theta - 2 \right) \]

The force of constraint in the radial direction is just the normal force from the surface. We get

\[ N = \frac{mg}{2} \left(3 \cos \theta - 2 \right) \]

\[ \Rightarrow \frac{\partial}{\partial r} N = \lambda = mg \left(3 \cos \theta - 2 \right) \]
We can now find out something interesting!

By definition, the normal force must be positive - i.e. pointing outward from surface.

But the expression we found

\[ N = mg (3 \cos \theta - 4) \]

will become negative when \( \cos \theta = \frac{2}{3} \),

i.e. when \( \theta = 48.2^\circ \).

What this tells us is that for \( \theta > 48.2^\circ \)
the normal force cannot provide what is needed to maintain the constraint
\[ \rightarrow \text{ the mass will fly off the surface!} \]
Rolling disk on hemispherical surface

disk has radius $a$

disk hemisphere has radius $R$

center of mass of disk

$\phi$ gives angle of rotation of disk

Constraint 1: disk rolls on surface

$g_1(r, \theta, \phi) = r - (R + a) = 0$

Constraint 2: disk rolls without slipping

$g_2(r, \theta, \phi) = a(\phi - \theta) - RE = 0$

(similar to problem of ball rolling inside cylinder)

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} I \dot{\phi}^2$

$U = m g r \cos \theta$

$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} I \dot{\phi}^2 - m g r \cos \theta$

$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 2 \frac{\partial g_1}{\partial r} + 2 \frac{\partial g_2}{\partial r}$

$\frac{\partial ^2 \mathcal{L}}{\partial r^2} = 0$

$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = m \ddot{r} + r \dot{\theta}^2 - m g \cos \theta$

$\ddot{\theta} = 0$

$\frac{\partial g_1}{\partial r} = 1$, $\frac{\partial g_2}{\partial r} = 0$
\( \Rightarrow \quad m \ddot{r} - m r \dot{\theta}^2 + m g \cos \theta = \lambda_1 \quad \) (1)

\( \frac{2 \ddot{r}}{\dot{\theta}} = m r^2 \dot{\theta}^2 \quad , \quad \frac{d}{dt} \left( \frac{2 \ddot{r}}{\dot{\theta}} \right) = m r^2 \dddot{\theta} + 2 m r \dot{r} \dot{\theta} \)

\( \frac{2 \ddot{r}}{\dot{\theta}} = m g r \sin \theta \quad , \quad \frac{\partial \delta_1}{\partial \theta} = 0 \quad , \quad \frac{\partial \delta_2}{\partial \theta} = -(r+a) \)

\( \Rightarrow \quad m r^2 \dddot{\theta} + 2 m r \dot{r} \dot{\theta} - m g r \sin \theta = -\lambda_2 (r+a) \quad \) (2)

\( \frac{2 \ddot{r}}{\dot{\phi}} = I \phi \quad , \quad \frac{d}{dt} \left( \frac{2 \ddot{r}}{\dot{\phi}} \right) = I \dddot{\phi} \quad , \quad \frac{\partial \delta_1}{\partial \phi} = 0 \quad , \quad \frac{\partial \delta_2}{\partial \phi} = a \)

\( \Rightarrow \quad I \dddot{\phi} = \lambda_2 a \quad \) (3)

From \( g_1 (r, \theta, \phi) \) we have \( r = r+a \)

\( \dot{r} = \dot{r} = 0 \)

(1) \( \Rightarrow \quad -m (r+a) \dot{\theta}^2 + m g \cos \theta = \lambda_1 \quad \) (4)

(2) \( \Rightarrow \quad m (r+a) \dddot{\theta} - m g (r+a) \sin \theta = -\lambda_2 (r+a) \)

\( \Rightarrow \quad m (r+a) \dddot{\theta} - m g \sin \theta = -\lambda_2 \quad \) (5)

From \( g_2 (r, \theta, \phi) = a (\phi - \phi) - k \theta = 0 \)

\( \Rightarrow \quad \phi = \frac{(r+a) \theta}{a} \quad \Rightarrow \quad \hat{\phi} = \frac{(r+a) \dot{\theta}}{a} \)

Substitute \( \dddot{\phi} \) (3) \( \Rightarrow \quad \dddot{\phi} = \frac{\lambda_2 a^2}{I (r+a)} \)
Substitute into (6) to get

\[ m \left( R + a \right) \frac{a^2}{I(R+a)} - mg \sin \theta = -a^2 \]

\[ a^2 \left( 1 + \frac{ma^2}{I} \right) = mg \sin \theta \]

\[ \Rightarrow a^2 = \frac{mg \sin \theta}{\left( 1 + \frac{ma^2}{I} \right)} \]

For a rolling disk \( I = \frac{1}{2} ma^2 \) \( \Rightarrow a^2 = \frac{1}{2} mg \sin \theta \)

Substitute above result for \( a^2 \) into

\[ \dot{\theta}^2 = \frac{a^2 a^2}{I(R+a)} = \frac{mg \sin \theta}{\left( 1 + \frac{ma^2}{I} \right)} \frac{a^2}{I(R+a)} = \frac{gs \sin \theta}{\left( 1 + \frac{I}{ma^2} \right)(R+a)} \]

as in previous example, \( \dot{\theta} = \dot{\theta} \frac{d \dot{\theta}}{d\theta} \) so

\[ \int \dot{\theta} \, d\theta = \frac{g}{\left( 1 + \frac{I}{ma^2} \right)(R+a)} \int \sin \theta \, d\theta \]

\[ \dot{\theta}^2 = \frac{2g}{\left( 1 + \frac{I}{ma^2} \right)(R+a)} (1 - \cos \theta) \]

where we assumed that \( \theta = 0 \) when \( \dot{\theta} = 0 \)
Substitute this into (4) gives

\[-m(R+a)\ddot{\theta} + mg\cos\theta = \lambda_1\]

\[\lambda_1 = -m(R+a) \frac{2g(1-\cos\theta)}{(1+\frac{I}{ma^2})(R+a)} + mg\cos\theta\]

\[
\lambda_1 = -\frac{2mg}{3(1-\cos\theta)} + mg\cos\theta
\]

for a disk with \(I = \frac{1}{2}ma^2\)

\[\lambda_1 = \frac{-4}{3} mg(1-\cos\theta) + mg\cos\theta\]

\[\lambda_1 = \frac{mg}{3}(2\cos\theta - 4)\]

Generalized forces:

in \(r\) direction, gives normal force

\[N = \lambda_1 \frac{\partial \phi}{\partial r} + \lambda_2 \frac{\partial \phi}{\partial \theta} = \lambda_1\]

we see that \(N\) becomes unphysically negative

when \(\cos\theta = \frac{4}{7}\), or when \(\theta = 55.15^\circ\)

(this is a larger angle than in previous example without rolling)

this would cause disk to leave surface

when \(\theta \geq 55.15^\circ\).
But another problem occurs before this happens in φ direction, the generalized force is the frictional torque

\[ T_\phi = F_\phi a = \frac{\lambda_1 \delta \theta}{\delta \phi} + \frac{\lambda_2 q_2}{\delta \phi} = \lambda_2 a \]

\[ F_\phi a = \frac{1}{3} \lambda m g \sin \theta \]

\[ \Rightarrow F_\phi = \frac{1}{3} m g \sin \theta \]

In θ direction, we get the same frictional torque

\[ T_\theta = \frac{\lambda_2 a}{\delta \theta} = \lambda_1 \frac{2q_1}{\delta \theta} + \lambda_2 q_2 - \lambda_2 (R+a) \]

Work done by \( T_\theta \) as disk moves \( \delta \theta \) should equal work done by \( T_\phi \) as disk moves \( \delta \phi \)

\[ T_\theta \delta \theta = \lambda_2 (R+a) \delta \theta = T_\phi \delta \phi = \lambda_2 a \delta \phi \]

\[ \Rightarrow (R+a) \delta \theta = a \delta \phi \]

but this is indeed true by constraint \( q_2 \)

Now, small disk rolls without slipping, the frictional force \( F_\phi \) is due to static friction.

We also have \( F_{\phi \text{max}} = \mu_s M \)

\[ \Rightarrow \mu_s = F_{\phi \text{max}} = \frac{\lambda_2 a}{\frac{1}{3} m g \sin \theta} \]

\[ \mu_s = \text{coefficient of static friction} \]

\[ \frac{1}{3} m g \left[ \cos \theta - \mu_{\text{max}} \right] = \frac{\lambda_2 a}{\frac{1}{3} m g \sin \theta \max} \]
If we define the ratio

\[ \mu = \frac{F_f}{N} \]

then we know that \( \mu \leq \mu_s \), the coefficient of static friction, \( F_{f_{\text{max}}} = \mu_s N \).

Now,

\[ \mu = \frac{F_f}{N} = \frac{\lambda_2}{A_1} = \frac{1}{3} \frac{mg \sin \theta}{\frac{mg}{3} (7 \cos \theta - 4)} \]

\[ \mu = \frac{\sin \theta}{7 \cos \theta - 4} \]

As \( \theta \) increases, \( \mu \) increases until it eventually reaches the value \( \mu_s \). For larger values of \( \theta \), the frictional force can no longer be large enough to maintain the no-slip constraint. The disk will start to slip as it falls down the hemispherical surface.

Note: The onset of slipping always occurs before the vanishing of the normal force, which causes the disk to leave the surface.

This is because \( \mu = \frac{F_f}{N} \), and \( \mu \) must always occur before \( N \to 0 \), assuming \( \mu_s \) is finite.
μ diverges at $\theta = 55.15^\circ$
where normal force vanishes

for $\mu_s = 1$, starts to slip at $\theta = 47.48^\circ$