General properties of motion

We had that the total mechanical energy was conserved

\[ E = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{\mu}{l^2} + U(r) \]

We can view this just like motion in one dimension, given by the coordinate \( r \), on an effective potential \( U_{\text{eff}}(r) = U(r) + \frac{1}{2} \frac{l^2}{\mu r^2} \).

For the gravitational force, \( U(r) = -\frac{l^2}{r} \), and we can graph the pieces as follows:

We see that \( U_{\text{eff}}(r) \) has a mininum value at some particular value of \( r \), that is a special place for which the force of \( U_{\text{eff}}(r) \) is attractive for \( U_{\text{eff}} \) to have \( \min U(r) \) must be attractive.

We can therefore use conservation of energy to classify the possible motions as follows:
i) For an energy $E_1 > 0$, there is a single turning point at $r_1$. This represents a particle coming in from $r = \infty$, scattering off the origin, and going back out to $r = \infty$.

This is an "unbound" state.

ii) For an energy $E_2 < 0 < E_3$, there are two turning points $r_2$ and $r_3$. The motion is confined between the radii $r_2$ and $r_3$.

This motion may be either as a closed orbit

or as an open orbit.

These are "bound" states.
iii) If $E_3 = U_{\text{min}}$ then the system must
stay at the potential minimum $r_3$, i.e. the
radius of motion $r(t)$ is constant in time.
This means a circular orbit

* $r_3$

iv) $E < U_{\text{min}}$ would require a negative
kinetic energy, and so is not allowed classically.

For the gravitational potential $U = \frac{-k}{r}$,

$$U_{\text{eff}} = -\frac{k}{r} + \frac{1}{2} \frac{L^2}{\mu r^2}$$

To its minimum at

$$\frac{\partial U_{\text{eff}}}{\partial r} = \frac{k}{r^2} - 2 \frac{L^2}{\mu r^4} = 0$$

$$\Rightarrow \mu kr = L^2 \Rightarrow r = \frac{L^2}{\mu k}$$

$$U_{\text{min}} = -\frac{k \mu k}{L^2} + \frac{1}{2} \frac{L^2}{\mu} \frac{L^2}{\mu L^2} = \frac{\mu k^2}{2 L^2}$$
For the bound state of case (ii) \( \text{min} < E < 0 \), we can determine whether the orbit is closed or open as follows:

we find earlier that

\[
\Delta \Theta = \frac{\pm \int dr \ell}{\mu r^2 \sqrt{\frac{2}{\mu} \left( E - U - \frac{\ell^2}{2\mu r^2} \right)}}
\]

First note that since \( \ell = \mu r^2 \hat{\theta} \) is constant, then \( \hat{\theta} \) can never change signs, therefore \( \Delta \Theta \) must always decrease or increase monotonically with time.

Now integrate the above going from \( r_{\text{min}} \) to \( r_{\text{max}} \) (the smallest and largest radial distances of the orbit) and then back from \( r_{\text{max}} \) to \( r_{\text{min}} \). The angular change in this is

\[
\Delta \Theta = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\ell dr}{\sqrt{\frac{2}{\mu} \left( E - U - \frac{\ell^2}{2\mu r^2} \right)}}
\]

(since change in \( \Theta \) from \( r_{\text{min}} \) to \( r_{\text{max}} \) is same as from \( r_{\text{max}} \) to \( r_{\text{min}} \))

If the \( \Delta \Theta \) is a rational fraction of \( 2\pi \), i.e.

\[
\Delta \Theta = \frac{2\pi a}{b} \quad \text{for integer } a \text{ and } b
\]
Then the orbit will close - after 6 periods of motion from $r_{mi}$ to $r_{mx}$ and back to $r_{mi}$, the particle will have made 6 revolutions about the origin.

**Planetary Motion**

$$U(r) = -\frac{k}{r}$$

We now focus on the problem where $U(r)$ is the gravitational potential $-k/r$.

We had

$$\Omega(r) = \int \frac{ldr}{r^2\sqrt{2\mu(E-U-E^2)}} + \text{const}$$

$$\Omega(r) = \int \frac{ldr}{r^2\sqrt{2\mu(E+\frac{k}{r}-\frac{E^2}{2\mu r^2})}} + \text{const}$$

Change variable of integration to $u = \frac{E}{r}$

$$du = \frac{E}{r^2} dr$$

$$\Omega = -\int du \frac{1}{\sqrt{2\mu(E+\frac{k}{u}-\frac{u^2}{2\mu})}} + \text{const}$$
\[ \theta = -\int \frac{du}{\sqrt{2\mu E + \frac{\mu k^2 u}{e} - u^2}} + \text{const} \]

Change integration variables to \( u = \frac{u - \mu k}{e} \quad du = du \)

\[ u^2 = u^2 - \frac{2\mu k u}{e} + \frac{\mu^2 k^2}{e^2} \frac{e^2}{e^2} \]

\[ \theta = -\int \frac{du}{\sqrt{2\mu E + \frac{\mu^2 k^2}{e^2} - u^2}} + \text{const} \]

Call \( c^2 = 2\mu E + \frac{\mu^2 k^2}{e^2} \frac{e^2}{e^2} \)

\[ \theta = -\int \frac{du}{\sqrt{c^2 - u^2}} + \text{const} \]

Can do this integral by trigonometric substitution

\[ \begin{array}{c}
\sqrt{c^2 - u^2} \\
\sin \phi = \frac{\sqrt{c^2 - u^2}}{c}
\end{array} \]

\[ \Rightarrow \cos \phi \, d\phi = -\frac{du}{c} \frac{1}{\sqrt{c^2 - u^2}} \Rightarrow d\phi = -\frac{du}{\sqrt{c^2 - u^2}} \]

\[ \theta = \int d\phi + \text{const} \]

Call the const \( \Theta_0 \)

\[ \Rightarrow \theta - \Theta_0 = \phi \]

\[ \cos (\theta - \Theta_0) = \cos \phi = \frac{u - \mu k/e}{c} = \frac{u - \mu k/e}{\sqrt{2\mu E + u^2 k^2 / e^2}} \]
\[
\cos (\theta - \theta_0) = \frac{U - \mu k r}{\mu k r} = \frac{\mu k r}{\sqrt{1 + \frac{2E}{\mu k^2}}} - 1
\]

\[
\cos (\theta - \theta_0) = \left( \frac{\frac{E^2}{\mu k^2 r} - 1}{\sqrt{1 + \frac{2E}{\mu k^2}}} \right)
\]

Note, when \( \theta = \theta_0 \), then

\[
\cos (\theta) = 1 = \frac{\frac{E^2}{\mu k^2 r} - 1}{\sqrt{1 + \frac{2E}{\mu k^2}}}
\]

\[
\Rightarrow \quad \frac{1 + \frac{2E}{\mu k^2}}{\mu k^2} = \left( \frac{\frac{E^2}{\mu k^2 r} - 1}{\sqrt{1 + \frac{2E}{\mu k^2}}} \right)^2 = \frac{\frac{E^4}{\mu^2 k^4 r^2} + 1 - \frac{2E^2}{\mu^2 k^2 r^2}}{\mu k^2}
\]

\[
\Rightarrow \quad E = \frac{\frac{E^2}{\mu k^2 r} - 1}{2\mu k^2 r} \quad \text{for energy when kinetic energy vanishes, i.e., when} \quad \theta = \theta_0
\]

So \( \theta = \theta_0 \) corresponds to \( r \) being one of the two turning points. As \( \theta \) increases above \( \theta_0 \), \( \cos (\theta - \theta_0) \) decreases, \( \Rightarrow \) \( r \) increases. Hence \( \theta = \theta_0 \) corresponds to the turning point at the smallest radial distance \( r_{\text{min}} \).
So solution is

$$\cos(\theta - \theta_0) = \left( \frac{\ell^2}{\mu k r} - 1 \right) \frac{\ell^2}{\sqrt{1 + 2E\ell^2}}$$

where \( \theta = \theta_0 \) gives \( r = r_{\text{min}} \).

Henceforth we choose \( \theta_0 = 0 \), so that \( \theta = 0 \).

At point of closest approach to the origin, \( r = r_{\text{min}} \).

If we define \( \alpha = \frac{\ell^2}{\mu k} \) and \( \varepsilon = \sqrt{1 + 2E\ell^2} \frac{\mu k^2}{\ell^2} \),

the solution is

$$\cos \theta = \frac{\alpha - 1}{\varepsilon}$$

$$\Rightarrow \frac{\alpha}{\ell} = 1 + \varepsilon \cos \theta$$

This is equation of conic section with one focus at the origin.

Conic sections are curves formed by intersection of a plane with a cone.
\[ \frac{x}{r} = 1 + E \cos \theta \quad E = \sqrt{1 + \frac{2E_0^2}{\mu k^2}} \]

For \( E = 0 \), i.e. \( E = -\frac{\mu k^2}{2E_0^2} \), we have a circular orbit.

For \( 0 < E < 1 \), i.e. \( E_{\text{min}} < E < 0 \), we have an elliptical orbit, \( E \) is the eccentricity.

For \( E = 1 \), i.e. \( E = 0 \), we have a parabolic orbit.

For \( E > 1 \), i.e. \( E > 0 \), we have a hyperbolic orbit.