When the coordinate axes are assumed to be given, one sometimes writes the vector \( \vec{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 \) as the triple of numbers \((x_1, x_2, x_3)\) as row vector or \((\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix})\) column vector.

The notation will be convenient when dealing with matrices.

**Rotation Matrix**

Suppose we know the coordinates \((x_1, x_2, x_3)\) of a vector with respect to a set of basis vectors \(\hat{e}_1, \hat{e}_2, \hat{e}_3\), i.e. \(\vec{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3\).

What will be the coordinates of \(\vec{r}' = (x'_1, x'_2, x'_3)\) with respect to a different set of basis vectors \(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3\) that are rotated with respect to the first?

Given \((x_1, x_2, x_3)\), how do we find \((x'_1, x'_2, x'_3)\)?

\[\hat{e}_1, \hat{e}_2\] are rotated by angle \(\theta\) with respect to \(\hat{e}'_1, \hat{e}'_2\).
Consider \( x'_i \) — this is the projection of \( \vec{r} \) onto unit vector \( \hat{e}_i \), so

\[
x'_i = \hat{e}_i \cdot \vec{r} = \hat{e}_i \cdot (x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3)
\]

\[
= (\hat{e}_i \cdot \hat{e}_1) x_1 + (\hat{e}_i \cdot \hat{e}_2) x_2 + (\hat{e}_i \cdot \hat{e}_3) x_3
\]

Similarly,

\[
x'_2 = (\hat{e}_2 \cdot \hat{e}_1) x_1 + (\hat{e}_2 \cdot \hat{e}_2) x_2 + (\hat{e}_2 \cdot \hat{e}_3) x_3
\]

\[
x'_3 = (\hat{e}_3 \cdot \hat{e}_1) x_1 + (\hat{e}_3 \cdot \hat{e}_2) x_2 + (\hat{e}_3 \cdot \hat{e}_3) x_3
\]

or, with summation notation

\[
x'_i = \sum_j (\hat{e}_i \cdot \hat{e}_j) x_j
\]

or, with summation notation:

\[
x'_i = \sum_j (\hat{e}_i \cdot \hat{e}_j) x_j
\]

(ambiguous)

\[\cos \angle \hat{e}_i \hat{e}_j = (\hat{e}_i \cdot \hat{e}_j)\]

(ambiguous)

In matrix notation:

\[
\begin{pmatrix}
x'_1 \\
x'_2 \\
x'_3
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

\[
x'_i = \sum_j a_{ij} x_j
\]

\[
\vec{r}' = \lambda \vec{r}
\]

\[\lambda \text{ is the rotation matrix.}\]
In our 2D example:

\[\mathbf{e}_1 = (\hat{e}_1' \cdot \hat{e}_1) = \cos \theta\]
\[\mathbf{e}_2 = (\hat{e}_2' \cdot \hat{e}_2) = \cos \theta\]
\[\mathbf{e}_3 = (\hat{e}_3' \cdot \hat{e}_2) = \cos (\frac{\pi}{2} - \theta) = \sin \theta\]
\[\mathbf{e}_4 = (\hat{e}_4' \cdot \hat{e}_1) = \cos (\frac{\pi}{2} + \theta) = -\sin \theta\]

Rotation matrix is:

\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

In a 3D rotation matrix, not all the nine \(a_{ij}\) are independent of each other. In fact, there are only 3 independent parameters.

To get the constraints that the \(a_{ij}\) satisfy, consider:

\[
\delta_{ij} = \hat{e}_i' \cdot \hat{e}_j
\]

Since \(\hat{e}_i\) are orthonormal basis vectors, we can write:

\[\hat{e}_i' = (\hat{e}_i' \cdot \hat{e}_1) \hat{e}_1 + (\hat{e}_i' \cdot \hat{e}_2) \hat{e}_2 + (\hat{e}_i' \cdot \hat{e}_3) \hat{e}_3\]

\[\Rightarrow \hat{e}_i' = \sum_k a_{ik} \hat{e}_k\]

Similarly:

\[\hat{e}_j' = \sum_m a_{jm} \hat{e}_m\]

So:

\[\hat{e}_i' \cdot \hat{e}_j' = \sum_{km} a_{ik} a_{jm} (\hat{e}_k \cdot \hat{e}_m)\]

\[= \sum_{km} a_{ik} a_{jm} s_{km}\]

\[\hat{e}_i' \cdot \hat{e}_j' = \sqrt{\sum_k a_{ik} a_{jk}} = \delta_{ij}\]
For \( i=j \), we get
\[ \lambda_{i1}^2 + \lambda_{i2}^2 + \lambda_{i3}^2 = 1 \]
This represents 3 equations corresponding to \( i=1, 2, 3 \).

For \( i \neq j \), we get
\[ \lambda_{i1} \lambda_{j1} + \lambda_{i2} \lambda_{j2} + \lambda_{i3} \lambda_{j3} = 0 \]
This represents 3 equations corresponding to
\( (i,j) = (1,3), (1,2), (2,3) \)

(Note: \( i=1, j=3 \) gives same equation as \( i=3, j=1 \).)

In total, we get 6 equations for the nine \( \lambda_{ij} \)
⇒ there are 3 free parameters describing the rotation.

One way to represent these three parameters is to say one has a rotation of angle \( \theta \) about an axis oriented in direction \( \hat{m} \). Since \( \hat{m} \) is a unit vector, \( m_1^2 + m_2^2 + m_3^2 = 1 \), so giving the direction \( \hat{m} \) uses 2 parameters, and giving the angle \( \theta \) uses the 3rd parameter.

The are other ways to parameterize the rotation
(Euler angles used in solid body rotations).
Inverse rotation

Suppose we knew the coordinates \((x'_1, x'_2, x'_3)\) with respect to basis \(\hat{e}_1', \hat{e}_2', \hat{e}_3'\). Then what are coordinates \((x_1, x_2, x_3)\) with respect to basis \(\hat{e}_1, \hat{e}_2, \hat{e}_3\)?

everything is the same as before except prime \(\Rightarrow\) unprimed

\[ \vec{r} = \vec{r}' \] where \(\vec{x}'_i = (\hat{e}_i \cdot \hat{e}'_i) = \lambda_{ij} \)

So if \(\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \)

\[
\begin{pmatrix}
\begin{array}{ccc}
\lambda_{11} & \lambda_{12} & \lambda_{13} \\
\lambda_{21} & \lambda_{22} & \lambda_{23} \\
\lambda_{31} & \lambda_{32} & \lambda_{33}
\end{array}
\end{pmatrix}
\]

then \(\vec{x}' = \begin{pmatrix} \lambda_{11} \lambda_{21} \lambda_{31} \\ \lambda_{12} \lambda_{22} \lambda_{32} \\ \lambda_{13} \lambda_{23} \lambda_{33} \end{pmatrix} \)

we say that \(\vec{x}'\) is the transposed matrix of \(\vec{x}\)

\[ \vec{x}' = \vec{x}^T \]

the transpose of any matrix \(\vec{A}\) with elements \(a_{ij}\) is the matrix \(\vec{A}^T\) with elements

\[(\vec{A}^T)_{ij} = a_{ji}\]

\((\vec{e}_i)\)th element of \(\vec{A}^T = (\vec{e}_j)\)th element of \(\vec{A}\)
Note also that \( \overrightarrow{X'} \) is the matrix of the inverse rotation of \( \overrightarrow{X} \). If one rotates by \( \overrightarrow{X} \) and then by \( \overrightarrow{X'} \), one winds up in the same original basis.

Algebraically:
\[
\overrightarrow{r} = \overrightarrow{X} \cdot \overrightarrow{r'} \quad \text{but} \quad \overrightarrow{r'} = \overrightarrow{X'} \cdot \overrightarrow{r}
\]

So \( \overrightarrow{r} = \overrightarrow{X} \cdot \overrightarrow{X'} \cdot \overrightarrow{r} = (\overrightarrow{X} \cdot \overrightarrow{X'}) \cdot \overrightarrow{r} \)

true for all \( \overrightarrow{r} \)
\[
\Rightarrow (\overrightarrow{X} \cdot \overrightarrow{X'}) = \overrightarrow{I} \quad \text{identity matrix}
\]
or in terms of indices:
\[
x_i' = \sum_j \lambda_{ij} x_j'
\]

and
\[
x_j' = \sum_k \lambda_{jk} x_k
\]

\[
\Rightarrow x_i = \sum_{jk} \lambda_{ij} \lambda_{jk} x_k
\]

above is true for all vectors \( \overrightarrow{r} \)
\[
\Rightarrow \sum_j \lambda_{ij} \lambda_{jk} = \delta_{ij}
\]

the inverse of a matrix \( \overrightarrow{X} \) is written as \( \overrightarrow{X}^{-1} \),
\[
\overrightarrow{X}^{-1} \overrightarrow{X} = \overrightarrow{A} \cdot \overrightarrow{A}^{-1} = \overrightarrow{I}
\]

So we conclude that \( \overrightarrow{X} = \overrightarrow{X}^{-1} \)

Combining with earlier result \( \overrightarrow{X}' = \overrightarrow{X}^{-1} \overrightarrow{X} \)
we get the very important result for rotation matrices that
\[ \theta^{-1} = -\theta \]
\[ \text{inverse} = \text{transpose} \]

Aside: assume you understand matrix multiplication
\[ C = \hat{A} \cdot \hat{B} \] means
\[ C_{ij} = \sum_k A_{ik} B_{kj} \]

\[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

Any matrix that satisfies the condition \( A^{-1} = A^T \) is said to be an orthogonal matrix.

Another property of orthogonal matrices is that of the determinant, then \( |A|^{-1} = 1 \)

This follows from the following:
For any matrix \( \hat{A} \), which has an inverse \( \hat{A}^{-1} \)
\[ |\hat{A}^{-1}| = \frac{1}{|A|} \] so \( |A||A^{-1}| = 1 \)

Also, for any matrix \( \hat{A} \), \( |A| = |A^T| \). For orthogonal, \( A^{-1} = A^T \), so \( |A||A^{-1}| = |A||A^T| = |A|^2 \)
Another property of orthogonal matrices:

If $A$ and $B$ are orthogonal, so is $C = AB$

Proof: $C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$ see HW

$$= B^T A^T$$ since $A, B$ orthogonal

$$(C^{-1})_{kj} = \sum_k B^T_{ck} (A^T)_{kj} = \sum_k B_{ki} A_{kj}$$

$$= \sum_k A_{ij} B_{kj} = C_{ij} = (C^T)_{ij}$$

So $C^{-1} = C^T$ and $C$ is orthogonal.

Physically this means that two successive rotations about different axes can always be viewed as a single rotation about a designated axis, some appropriate other axis.
Proper and Improper Rotations

For an orthogonal matrix \( \hat{A} \), \( |\hat{A}| = 1 \)

If \( |\hat{A}| = 1 \) we say it is a proper rotation.
If \( |\hat{A}| = -1 \) we say it is an improper rotation.

Example of improper rotation is inversion:

\[
\hat{A} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]
\( \hat{e}_0' = -\hat{e}_0 \)

turns a right-handed coordinate basis into a left-handed coordinate basis.

All orthogonal matrices obtained by a series of ordinary rotations have determinant = 1.
All orthogonal matrices which are improper rotations can be written as a product of proper rotations times an inversion.

Under inversion, the components of a vector are reflected:

\[
\tilde{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3
= -A_1 \hat{e}_1' - A_2 \hat{e}_2' - A_3 \hat{e}_3'
\]
\( \text{inverted basis} \)

So \( A_1' = -A_1 \)

Consider cross product:

\[
\tilde{C} = \hat{A} \times \hat{B} \]

under inversion

\[
C_i = \sum_k \epsilon_{ijk} A_j B_k
\]
Under inversion

\[ C_i' = \sum_k \varepsilon_{ijk} A_j' B_k' = \sum_k \varepsilon_{ijk} (A_j')(B_k') \]

\[ = \sum_k \varepsilon_{ijk} A_j B_k = C_i \]

\[ C_i' = C_i \]

We say \( \vec{C} \) is a pseudo vector - under an inversion, its coordinates are not reflected.

The laws of physics must be written so that all terms in an equation are either vectors or pseudo-vectors.

Example: Lorentz force \( \vec{F} = q \vec{E} + q \vec{v} \times \vec{B} \)

\( \vec{F} \) and \( \vec{v} \) are vectors \( \rightarrow \vec{E} \) is a vector

\( \vec{B} \) is a pseudo-vector

(\text{Since vector} \times \text{pseudo} = \text{vector})
Finite and Infinite Rotations

A rotation can be described by 3 parameters. Since a vector also has 3 parameters, one might think that a rotation can be represented by a vector. For example, we might try to represent a rotation through angle $\theta$ about the axis of rotation $\hat{m}$ by a vector $\vec{v} = \theta \hat{m}$.

\[ \text{A second rotation through angle } \theta' \text{ about axis } \hat{m}' \text{ would be described by a vector } \vec{v}' = \theta' \hat{m}'. \]

But if this idea of a rotation vector makes sense, one would want that a rotation $\vec{v}$, followed by a rotation $\vec{v}'$, would be described by the vector $\vec{v} + \vec{v}'$.

But since vector addition is commutative, $\vec{v} + \vec{v}' = \vec{v}' + \vec{v}$, this would imply that rotations commute. But this cannot be so, because rotations are given by matrices and matrices do not in general commute.

Example: consider rotation by 90° about $\vec{e}_1$, followed by 90° rotation about $\vec{e}_2$. This is not the same as rotation by 90° about $\vec{e}_2$, followed by a 90° rotation about $\vec{e}_1$. 
However, the above problem does not occur for infinitesimal rotations. Infinitesimal rotations do commute. This fact allows us to define the angular velocity vector $\omega$.

Consider infinitesimal rotation through angle $\delta \theta$ about rotation axis $\hat{m}$, each infinitesimal rotation $\delta \theta$.

$$\hat{m} \times \vec{r} \quad \text{change in } \vec{r} \quad \text{above rotation is}$$

$$\delta \vec{r} = r \sin \phi \delta \theta$$

direction of $\delta \vec{r}$ is direction of $\hat{m} \times \vec{r}$

$$\Rightarrow \delta \vec{r} = \delta \theta \hat{m} \times \vec{r}$$

where $\delta \vec{r} = \delta \theta \hat{m}$ is the vector describing the infinitesimal rotation.

Now consider a second rotation $\delta \theta' = \delta \theta' \hat{m}$.
After the 1st rotation, our vector \( \vec{r} \) is transformed into the vector \( \vec{r} + \vec{\theta} \times \vec{r} \).

After the 2nd rotation, the vector is transformed into

\[
(\vec{r} + \vec{\theta} \times \vec{r}) + \vec{\theta}' \times (\vec{r} + \vec{\theta} \times \vec{r})
\]

\[
= \vec{r} + \vec{\theta} \times \vec{r} + \vec{\theta}' \times \vec{r} + \vec{\theta}' \times (\vec{\theta} \times \vec{r})
\]

2nd order in infinitesimals \( \Rightarrow \) ignore

So we end up at vector

\[
\vec{r} + \vec{\theta} \times \vec{r} + \vec{\theta}' \times \vec{r} = \vec{r} + (\vec{\theta} + \vec{\theta}') \times \vec{r}
\]

If we now rotate by \( \vec{\theta}' \) first, we get \( \vec{r} + \vec{\theta}' \times \vec{r} \).

Then if we next rotate by \( \vec{\theta} \) we get

\[
(\vec{r} + \vec{\theta}' \times \vec{r}) + \vec{\theta} \times (\vec{r} + \vec{\theta}' \times \vec{r})
\]

\[
= \vec{r} + \vec{\theta}' \times \vec{r} + \vec{\theta} \times \vec{r} + \vec{\theta} \times (\vec{\theta}' \times \vec{r})
\]

2nd order \( \Rightarrow \) ignore

\[
\Rightarrow \vec{r} + (\vec{\theta} + \vec{\theta}') \times \vec{r}
\]

So the two infinitesimal rotations do commute.

Moreover, the product of the two rotations looks just like a single infinitesimal rotation with rotation vector \( \vec{\theta}' + \vec{\theta} \) as expected if the infinitesimal rotation is indeed describable as a vector.
For a rotating solid body, the change in position of a point at position \( \vec{r} \), after an infinitesimal rotation by \( \delta \vec{\theta} \) is

\[ \delta \vec{r} = \delta \vec{\theta} \times \vec{r} \]

If this rotation takes place in a time \( \delta t \), then we get for the velocity of the point at \( \vec{r} \),

\[ \vec{v} = \frac{\delta \vec{r}}{\delta t} = \frac{\delta \vec{\theta}}{\delta t} \times \vec{r} \]

We define the angular instantaneous angular velocity of the solid body as

\[ \vec{\omega} = \frac{\delta \vec{\theta}}{\delta t} \]

and then

\[ \vec{v} = \vec{\omega} \times \vec{r} \]

The fact that \( \delta \vec{\theta} \) is a vector \( \Rightarrow \) angular velocity \( \vec{\omega} \) is a vector \( \Rightarrow \) points along the instantaneous axis of rotation.

For a point particle traveling on a trajectory \( \vec{r}(t) \), we can also define an instantaneous angular velocity as follows. Consider the plane spanned by \( \vec{v}(t) = \vec{r}(t) \) and \( \vec{\alpha}(t) = \vec{r}'(t) \). Broadly, the particle is moving in that plane, on a path with an instantaneous reader of curvature.
The plane containing \( \mathbf{v} \) and \( \mathbf{a} \) at time \( t \) is denoted as \( \mathbf{r}(t) \).

\[ R = \text{radius of curvature at point } \mathbf{r}(t) \text{ in the plane of } \mathbf{r}(t) \]

Instantaneously, the particle looks as if it is going in a circle of radius \( R \) in the plane spanned by \( \mathbf{v} \) and \( \mathbf{a} \). An axis through point "O", perpendicular to the plane, is the instantaneous axis of rotation. If the particle moves an angular distance \( \theta \) in time \( \Delta t \), the instantaneous angular velocity is \( \bar{\omega} = \frac{\Delta \theta}{\Delta t} \), where \( \bar{\omega} \) points along the instantaneous axis of rotation.

Then, \( \bar{\omega} = \bar{\omega} \times \mathbf{r} \).