If we have computed \( \mathbf{T} \) in one coord system, how can we find its values in another coord system?

1. **Parallel Axes Theorem** - translation of coord axes

   Consider two coordinate systems, \( xyz \) and \( x'y'z' \), which have the same orientation of basis vectors, but are separated by a displacement \( \mathbf{R} \).

   A point \( P \) is given by the vector \( \mathbf{r} \) in the \( xyz \) system, and by the vector \( \mathbf{r}' = \mathbf{r} - \mathbf{R} \) in the \( x'y'z' \) coord system.

   In the \( xyz \) system
   \[
   I_{ij} = \sum_a m_a \left[ r_a^2 \delta_{ij} - x_{ai} x_{aj} \right]
   \]

   In the \( x'y'z' \) system
   \[
   I'_{ij} = \sum_a m_a \left[ r_a'^2 \delta_{ij} - x'_{ai} x'_{aj} \right]
   \]

   Substitute
   \[
   r_a'^2 = (\mathbf{r}_a - \mathbf{R})^2 = r_a^2 + \mathbf{R}^2 - 2\mathbf{r}_a \mathbf{R}
   \]

   \[x'_{ai} x'_{aj} = (x_{ai} - R_i)(x_{aj} - R_j) = x_{ai} x_{aj} + R_i R_j - x_{ai} R_j - x_{aj} R_i\]

   \[\Rightarrow I'_{ij} = \sum_a m_a \left[ r_a'^2 \delta_{ij} - x_{ai} x_{aj} \right] + \sum_a m_a \left[ \delta_{ij} (\mathbf{R}^2 - 2\mathbf{r}_a \mathbf{R}) - R_i R_j + x_{ai} R_j - x_{aj} R_i \right]\]

   \[= I_{ij} + \delta_{ij} \left[ M \mathbf{R}^2 - 2 \left( \sum_a m_a \mathbf{r}_a \right) \mathbf{R} \right] - M R_i R_j + \sum_a m_a x_{ai} R_j + \sum_a m_a x_{aj} R_i\]
If we choose \( x', y', z' \) to be the center of mass coordinate system, i.e., the origin of \( x', y', z' \) at the center of mass of the body, then \( \bar{R} \) is the center of mass position in the mutual fixed frame, and \( \sum \alpha_m x_{\alpha c} = M \bar{R} \) for \( I_{ij} = I_{ij} + M \left( \delta_{ij} \bar{R}^2 - R_i R_j \right) - 2MR^2S_{ij} + MR_i R_j + MR_j R_i \)

\[ I_{ij}' = I_{ij} - M \left( \delta_{ij} \bar{R}^2 - R_i R_j \right) \]

Parallel axis theorem relates \( I \) in the center of mass frame to \( I \) in a displaced frame.

\( I_{ij} = I_{ij}' + M \left( \delta_{ij} \bar{R}^2 - R_i R_j \right) \)

where \( \bar{R} \) is location of origin of primed coord system as measured in unprimed coord syst. 
\( \bar{R} \) is also location of CM.

**Example:** we computed \( I_{ij} \) for a solid cube with coords centered on one corner. What is \( I_{ij}' \) for coords centered at the center of the cube?

![Diagram of a cube with axes labeled x, y, z.]  

Cube's center is at \( \bar{R} = \frac{1}{2} \left( \frac{x}{4} + \frac{y}{4} + \frac{z}{4} \right) \) with respect to the unprimed coord system.

\[ R^2 = \frac{b^2}{4} \cdot 3\]

\[ I_{ij}' = I_{ij} - M \left( \delta_{ij} \bar{R}^2 - R_i R_j \right) = \]

\[ = Mb^2 \left( \begin{array}{ccc} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{array} \right) - Mb^2 \left( \begin{array}{ccc} 1/2 & -1/4 & -1/4 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & -1/4 & 1/2 \end{array} \right) \]
\[ I'_{ij} = M_b^2 \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} = \frac{M_b^2}{6} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\( I'_{ij} \) is diagonal in the primed coord system.

With respect to the origin at the center of the cube, the inertia tensor is \( \bar{I}' = \frac{M_b^2}{6} \) on the identity tensor.

All principle moments of inertia (eigenvectors) are equal \( \Rightarrow \bar{I}' \) in diagonal in any coord system whose origin is at the center of the cube.

Angular momentum about center of cube will be

\[ \bar{L}' = \bar{I}' \cdot \bar{\omega} = \frac{M_b^2}{6} \bar{\omega} \]

\( \bar{L}' \) always parallel to \( \bar{\omega} \) no matter what is the direction of \( \bar{\omega} \).

Moral: The principle moments of inertia depend on where you choose to put the origin of the coord system.
(2) Rotation of coord axes

If know \( I_{ij} \) in the \( xyz \) system, what is \( I_{ij} \) in the \( x'y'z' \) system?

\[
I'_{ij} = \hat{e}'_i \cdot I \cdot \hat{e}'_j
\]

We can always expand the \( \hat{e}'_i \) in the basis vectors \( \hat{e}_j \)

\[
\hat{e}'_i = \sum_l (\hat{e}'_i \cdot \hat{e}_l) \hat{e}_l
\]

\[
\hat{e}'_j = \sum_k (\hat{e}'_j \cdot \hat{e}_k) \hat{e}_k
\]

So

\[
I'_{ij} = \hat{e}'_i \cdot I \cdot \hat{e}'_j = \sum_{l,k} (\hat{e}'_i \cdot \hat{e}_l) \hat{e}_l \cdot I \cdot \hat{e}_k (\hat{e}'_j \cdot \hat{e}_k)
\]

\[
= \sum_{l,k} (\hat{e}'_i \cdot \hat{e}_l) I_{lk} (\hat{e}'_k \cdot \hat{e}'_j)
\]

Now recall that \((\hat{e}'_i \cdot \hat{e}_k) = \lambda_{ik}\) is the rotation matrix from the unprimed system to the primed system.

\[
(\hat{e}'_i \cdot \hat{e}'_j) = \lambda_{ij} = (\lambda^t)_{kj}
\]

\(\lambda^t\) is transpose of rotation matrix

\[
I'_{ij} = \sum_{l,k} \lambda_{ik} I_{lk} \lambda^t_{kj} = \lambda \cdot I \cdot \lambda^t
\]

Matrix multiplication

The above rule is how tensors transform under rotation.
Transform under rotation

**Vector:** \[ \mathbf{v}'_i = \sum_j a_{ij} \mathbf{v}_j \]

**Tensor:** \[ I'_{ij} = \sum_{lk} a_{il} I_{lk} a^*_{kj} \]

\[ \alpha = \sum_{ek} I_{ek} a_{il} a_{jk} \]
We want to find dynamic equations that describe how the solid body rotates when an external torque is applied. This will come from

\[
\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}_{\text{fix}}
\]

\(\vec{\tau}_{\text{ext}}\) is the total external torque applied to the body,

\(\frac{d\vec{L}}{dt}_{\text{fix}}\) is the rate of change of the angular momentum, as seen in the fixed inertial frame of reference.

(This equation holds in inertial frames.)

We can now write

\[
\frac{d\vec{L}}{dt}_{\text{fix}} = \frac{d\vec{L}}{dt}_{\text{rot}} + \vec{\omega} \times \vec{L}
\]

where \(\frac{d\vec{L}}{dt}_{\text{rot}}\) is the rate of change of angular momentum as seen in the body frame.

\[
\vec{L} = \vec{I} \cdot \vec{\omega}
\]

Since \(\vec{L}\) is computed in the body frame, it is independent of the body frame.

\[
\frac{d\vec{L}}{dt}_{\text{rot}} = \frac{d(\vec{I} \cdot \vec{\omega})}{dt}_{\text{rot}} = \vec{I} \cdot \frac{d\vec{\omega}}{dt}_{\text{rot}} = \vec{I} \cdot \dot{\vec{\omega}}
\]

(remember \(\frac{d\vec{L}}{dt}_{\text{rot}} = (\frac{d\vec{L}}{dt}_{\text{fix}})_{\text{rot}}\), so we just write \(\dot{\vec{\omega}}\) for both.

\[
\vec{\omega} \times \vec{L} = \vec{\omega} \times (\vec{I} \cdot \vec{\omega})
\]

\[
\frac{d\vec{L}}{dt}_{\text{fix}} = \vec{I} \cdot \dot{\vec{\omega}} + \vec{\omega} \times (\vec{I} \cdot \vec{\omega})
\]

If we use as the coordinate axes of the body frame the principle axes of rotation, then
\[
\omega = \begin{pmatrix}
I_1 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_3
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} = 
\begin{pmatrix}
I_1 \omega_1 \\
I_2 \omega_2 \\
I_3 \omega_3
\end{pmatrix}
\]

\[
\omega \times (I \cdot \omega) = 
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} \times 
\begin{pmatrix}
I_1 \omega_1 \\
I_2 \omega_2 \\
I_3 \omega_3
\end{pmatrix} = 
\begin{pmatrix}
I_3 \omega_3 \omega_2 - I_2 \omega_2 \omega_3 \\
I_1 \omega_1 \omega_3 - I_3 \omega_3 \omega_1 \\
I_2 \omega_2 \omega_1 - I_1 \omega_1 \omega_2
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
(I_3 - I_2) \omega_3 \omega_2 \\
(I_1 - I_3) \omega_1 \omega_3 \\
(I_2 - I_1) \omega_2 \omega_1
\end{pmatrix}
\]

Combining the pieces with \( \vec{N}^{ext} = \vec{\omega} \times (I \cdot \vec{\omega}) + \vec{I} \cdot \vec{\omega} \) we get

\[
\begin{cases}
I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_3 \omega_2 = N_1^{ext} \\
I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = N_2^{ext} \\
I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_2 \omega_1 = N_3^{ext}
\end{cases}
\]

When no external torque is applied, the above Euler's equations for force-free rotation:

\[
\begin{cases}
I_1 \dot{\omega}_1 - (I_3 - I_2) \omega_3 \omega_2 = 0 \\
I_2 \dot{\omega}_2 - (I_1 - I_3) \omega_1 \omega_3 = 0 \\
I_3 \dot{\omega}_3 - (I_2 - I_1) \omega_2 \omega_1 = 0
\end{cases}
\]
Stability of force-free solid body rotations

We want to look at the stability of force free rotations of a solid body about its principle axes of inertia.

Suppose a body is rotating about one of the principle axes of inertia, for example \( \hat{e}_1 \).

\[ \vec{\omega} = \omega_1 \hat{e}_1 \]

We now ask what happens if \( \vec{\omega} \) has a small perturbation in the \( \hat{e}_2 \) or \( \hat{e}_3 \) directions.

\[ \vec{\omega} = \omega_1 \hat{e}_1 + \delta \omega_2 \hat{e}_2 + \delta \omega_3 \hat{e}_3 \quad \delta \omega_2 \ll \omega_1, \delta \omega_3 \ll \omega_1 \]

Euler's equations become:

1) \( I_1 \dot{\omega}_1 + (I_3 - I_2) \delta \omega_2 \delta \omega_3 = I_1 \dot{\omega}_1 = 0 \)
2) \( I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \delta \omega_3 = 0 \)
3) \( I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \delta \omega_2 = 0 \)

In the first line we set \( \delta \omega_2 \delta \omega_3 = 0 \), since this is second order in the perturbation. Here we are doing a linear stability analysis, i.e. keeping only the leading (linear) terms in the perturbation.
1) \( \omega_i = 0 \) or \( \omega_i = \text{constant} \)

2) \( \ddot{S}w_2 + a\dot{S}w_3 = 0 \) where \( a = \left( \frac{I_1 - I_3}{I_2} \right) \omega_i \)

3) \( \ddot{S}w_3 = b \dot{S}w_2 \) where \( b = \left( \frac{I_1 - I_2}{I_3} \right) \omega_i \)

2) \( \ddot{S}w_k + a\dot{S}w_3 = 0 \)

Substitute from 3) \( \Rightarrow \ddot{S}w_2 + ab \dot{S}w_2 = 0 \)

Similarly \( \ddot{S}w_3 + ab \dot{S}w_3 = 0 \)

The above look just like equation for simple harmonic oscillator, provided \( ab \) is a positive constant. If \( ab > 0 \), then \( \dot{S}w_2 \) and \( \dot{S}w_3 \) will oscillate about zero with an angular frequency of

\[
\Omega = \sqrt{ab} = \sqrt{\left( \frac{I_1 - I_3}{I_2} \right) \left( \frac{I_1 - I_2}{I_3} \right) \omega_i}
\]

The notation about the \( \hat{z} \) axis will be stable.

If, however, \( ab < 0 \), then the solution to

\( \ddot{S}w_2 + ab \dot{S}w_2 = 0 \)

is of the form

\[
\dot{S}w_2(t) = A e^{-\lambda t} + B e^{\lambda t} \quad \text{with} \quad \lambda = \sqrt{|ab|}
\]

and the second term will cause the perturbation to grow exponentially — the notation about \( \hat{z} \) axis will be unstable.
New:
\[ ab = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3} \omega^2 \]

If \( I_1 \) is the largest of all the principal moments of inertia, then \( (I_1 - I_3)(I_1 - I_2) > 0 \) and so \( ab > 0 \), rotation is stable.

If \( I_1 \) is the smallest of all the principal moments of inertia, then \( (I_1 - I_3)(I_1 - I_2) < 0 \), but \( (I_1 - I_3)(I_1 - I_2) > 0 \), so \( ab > 0 \), rotation is stable.

If \( I_1 \) is the middle of the three values of the principal moments of inertia, then \( (I_1 - I_3)(I_1 - I_2) < 0 \), so \( ab < 0 \), rotation is unstable.

Conclusion: Rotation about principal axis corresponding to the largest and the smallest principal moments is always stable. Rotation about the principal axis corresponding to the middle moment is always unstable.