**Newton's 1st Law**

A body at rest or in uniform motion continues so unless acted on by external force.

**Newton's 2nd Law**

\[ F = ma = m \frac{d\vec{v}}{dt} = \dot{\vec{p}} \]

where momentum \( \vec{p} = m \vec{v} \)

\[ \Rightarrow \text{says what a force} \]

\[ \Rightarrow \text{defines what mass} \]

**Newton's 3rd Law**

If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction. If \( \vec{F}_1 \) is force on particle 1 from particle 2, and \( \vec{F}_2 \) is force on particle 2 from particle 1, then

\[ \vec{F}_1 = -\vec{F}_2 \]

\[ \Rightarrow \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \Rightarrow \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \]

\[ \Rightarrow \vec{p}_1 + \vec{p}_2 = \text{constant} \]

\[ \Rightarrow \frac{d}{dt} (\vec{p}_1 + \vec{p}_2) = 0 \text{ momentum conservation} \]

\( \text{(similar argument works for any number of interacting particles)} \)
Mass

Newton's 2nd law; mass is defined as the ratio of the applied force \( F \) to the resultant acceleration \( a \):

\[
\frac{|F|}{|a|} = m
\]

This is sometimes called the inertial mass - the resistance to acceleration or to change in inertia.

If there is a standard force, \( F_0 \), say, a spring, and a standard mass \( m_0 \), then one can determine the mass of any other object relative to \( m_0 \). If \( a_0 = \frac{F_0}{m_0} \) is the measured acceleration of the standard mass \( m_0 \) when acted on by the standard force \( F_0 \), and if \( a \) is the measured acceleration of another object when acted on by \( F_0 \), then the mass of the object is

\[
m = \frac{F_0}{a} = \frac{a_0 m_0}{a} = (\frac{a_0}{a})m_0
\]

One well known force is the force of gravity at the surface of the earth. The gravitational force is assumed to be \( \vec{F}_g = mg \hat{\mathbf{y}} \), where \( g = 9.8 \text{ m/s}^2 \) downwards. The gravitational mass \( m_g \) is what you measure when you weigh an object on a by compressing a spring in a scale. The acceleration \( \vec{a} \) of an object in the gravitational force is \( \vec{a} = \frac{\vec{F}_g}{m} = \left(\frac{m_g}{m}\right) \hat{\mathbf{y}} \).
The principle of equivalence was first shown by Galileo. It is that $m = mg$ for all objects. Inertial mass (the coefficient that determines acceleration in Newton's 2nd Law) equals the gravitational mass (the coefficient that determines the strength of the gravitational force acting on a body).

**Inertial Frame of Reference**

Reference frame in which Newton's laws of motion are valid — it is a non-accelerating frame of reference.

Since $F = ma = m \frac{d^2\vec{r}}{dt^2}$,

if one makes a change of coordinates $\vec{r}' = \vec{r} - \vec{u}t$ to a frame of reference moving with constant velocity $\vec{u}$, then

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{r}}{dt^2} = F$$

So Newton's laws hold also in the moving frame. This is called Galilean invariance.
Work and Energy

The work done by a force \( \vec{F} \) on a particle as the particle moves from position \( \vec{r}_1 \) to position \( \vec{r}_2 \) is

\[
W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \quad \text{line integral}
\]

If \( \vec{F} \) is the total force on the particle, then

\[
\vec{F} \cdot d\vec{r} = (m \ddot{\vec{r}}) \cdot \left( \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right) \right) = (m \ddot{\vec{r}}) \cdot \left( \frac{d\vec{u}}{dt} \right)
\]

\[
= m \frac{d}{dt} \left( \frac{1}{2} \vec{u}^2 \right) \quad \text{(Since } \frac{d}{dt} \vec{v} = \frac{d}{dt} \left( \frac{1}{2} \vec{v}^2 \right) = \frac{1}{2} \vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \frac{d\vec{v}}{dt} \text{)}
\]

\[
\vec{F} \cdot d\vec{r} = \frac{d}{dt} \left( \frac{m}{2} \vec{u}^2 \right) dt = \frac{d}{dt} \left( \frac{m}{2} \vec{u}^2 \right)
\]

\[
W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{d}{dt} \left( \frac{m}{2} \vec{u}^2 \right) dt = \int_{\vec{r}_1}^{\vec{r}_2} \frac{d}{dt} \left( \frac{m}{2} \vec{u}^2 \right) dt = \frac{m}{2} \vec{u}_2^2 - \frac{m}{2} \vec{u}_1^2
\]

Define \( T = \frac{1}{2} m \vec{u}^2 \) kinetic energy

work done = change in kinetic energy

\[
W_{12} = T_2 - T_1
\]
If $W_{12} > 0$, work done on particle is positive. 

Then $T_2 > T_1$ \(\Rightarrow\) kinetic energy increases.

**Conservative forces**

In general, $\int F \cdot dr$ may depend on the path taken in going from "1" to "2".

![Path diagram](image)

For certain forces, however, this integral is independent of the path. Such forces are called **conservative forces**.

Conservative forces are forces that can be written as the gradient of a scalar field $U$.

$$\vec{F} = -\nabla U$$

Then $W_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{r} = -\int_{1}^{2} \nabla U \cdot d\vec{r} = -\int_{1}^{2} du$

$$W_{12} = -(U(r_2) - U(r_1)) = U_1 - U_2$$

$U$ is called the **potential energy** of the force $\vec{F}$. 
For conservative force, \( \oint \mathbf{F} \cdot d\mathbf{r} = 0 \) for closed loop

Stokes theorem: \( \oint \mathbf{F} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{F}) \cdot d\mathbf{a} \)

\( \nabla \times \mathbf{F} = 0 \) for conservative force can check that it is always true that

\( \nabla \times (\nabla U) = 0 \) for any scalar function \( U \).

Now \( W_{12} = T_2 - T_1 = U_1 - U_2 \)

\( \Rightarrow T_2 + U_2 = T_1 + U_1 \equiv E \) total mechanical energy

\( \Rightarrow \) Conservation of mechanical energy when forces are conservative

Example: gravity at Earth's surface \( \mathbf{F}_g = m \ddot{\mathbf{g}} = -mg \hat{z} \).

where \( \hat{z} \) is normal to surface term

\( \mathbf{F} = -\nabla U \) where \( U = mgz \)

Note: potential energy is not absolute - can always add any constant to the function \( U \), without changing the force \( \mathbf{F} = -\nabla U \).

Also, \( T \) is not absolute, since can always transform to another moving inertial frame, energy conservation which would change velocity \( \dot{\mathbf{r}} \).
**Conservation Theorems**

**Linear momentum**

When total force \( \vec{F} = 0 \), then

\[
\vec{F} = \frac{d\vec{p}}{dt} = 0 \quad \text{and} \quad \vec{F} = \text{constant momentum conserved}
\]

If the component of \( \vec{F} \) in any particular direction \( \hat{e} \) vanishes, then

\[
\vec{F} \cdot \hat{e} = \frac{d\vec{p}}{dt} \cdot \hat{e} = 0 \quad \Rightarrow \quad \vec{p} \cdot \hat{e} = \text{constant}
\]

Component of momentum in direction \( \hat{e} \) is conserved.

Example: in gravitational field \( \vec{F}_g \), components of \( \vec{F} \) parallel to surface of earth are zero \( \Rightarrow \) horizontal components of momentum are conserved.

**Angular momentum**

Angular momentum of a particle with respect to an origin from which the position \( \vec{r} \) is measured is defined to be

\[
\vec{L} = \vec{r} \times \vec{p}
\]

The torque with respect to the same origin is

\[
\vec{\tau} = \vec{r} \times \vec{F}
\]
\[ \vec{I} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \]

Consider
\[ \frac{d\vec{I}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{F}) = \frac{d\vec{r}}{dt} \times \vec{F} + \vec{r} \times \frac{d\vec{F}}{dt} \]

But
\[ \frac{d\vec{r}}{dt} \times \vec{F} = \vec{v} \times m\vec{a} = 0 \text{ as } \vec{v} \times \vec{v} = 0 \]

So
\[ \frac{d\vec{I}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{\tau} \]

Angular version of \[ \vec{F} = \frac{d\vec{p}}{dt} \]

If component of torque in a particular direction \( \hat{e} \) vanishes, \( \vec{\tau} \cdot \hat{e} = 0 \), then
\[ \vec{\tau} \cdot \hat{e} \frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{\tau} \cdot \vec{L}) = 0 \]

then the component of \( \vec{L} \) in the direction \( \hat{e} \) is conserved
\[ \vec{L} \cdot \hat{e} = \text{constant} \]
Energy

- Generally, the dependent potential energy is

\[ E = T + U \]

Total mechanical energy

\[ \frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} \]

Potential energy for any conservative forces

\[ W = F \cdot d\vec{r} = dT \Rightarrow \frac{dT}{dt} = F \cdot \frac{d\vec{r}}{dt} \]

\[ \frac{dU(\vec{r}(t), t)}{dt} = \sum \frac{\partial U}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial U}{\partial t} = (\nabla U) \cdot \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t} \]

\[ \Rightarrow \frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = F \cdot \frac{d\vec{r}}{dt} + (\nabla U) \cdot \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t} \]

\[ = (F + \nabla U) \cdot \frac{d\vec{r}}{dt} + \frac{\partial U}{\partial t} \]

If a force is conservative then \( F = -\nabla U \) then

\[ \frac{dE}{dt} = \frac{dU}{dt} \]

If potential energy \( U \) is not an explicit function of time \( t \), then \( \frac{dU}{dt} = 0 \) \( \Rightarrow \frac{dE}{dt} = 0 \) and

mechanical energy is conserved.

In earlier derivation, it should have been

\[ W_{12} = \int \vec{F} \cdot d\vec{r} = -\int \nabla U(\vec{r}(t), t) \cdot \frac{d\vec{r}}{dt} \, dt = -\int \left[ \frac{dU}{dt} - \frac{\partial U}{\partial t} \right] \, dt \]

\[ W_{12} = -\int dU + \int \frac{\partial U}{\partial t} \, dt = U(r_1, t_1) - U(r_2, t_2) + \int \frac{\partial U}{\partial t} \, dt \]