Applications of Newton's Laws

\[ \vec{F} = \frac{d\vec{p}}{dt} = m \frac{d\vec{v}}{dt} = m \frac{d^2\vec{r}}{dt^2} \]

Solve second order differential equation

If one knows the force \( F(t) \) as function of time, then can solve equation of motion by integration of given initial conditions \( \vec{r}(0) \) and \( \vec{v}(0) \)

\[ \vec{v}(t) = \int_{0}^{t} \frac{\vec{F}(t)}{m} \, dt + \vec{v}(0) \]

\[ \vec{r}(t) = \int_{0}^{t} \vec{v}(t) \, dt + \vec{r}(0) \]

In general, not so simple since \( \vec{F} \) may depend on position of particle \( \vec{r} \).

\[ \text{Constant force problems} \]

Constant force \( \Rightarrow \) constant acceleration

\[ \vec{F} = \vec{F}_{g} + \vec{N} + \vec{F}_{f} = m\vec{a} \]

Take components perpendicular \& parallel to surface
As a simple case, consider a constant force $\vec{F}$. 

$$\vec{U}(t) = \int_0^t \vec{F} \frac{dt}{m} + \vec{U}(0) = \frac{\vec{F}}{m} t + \vec{U}(0)$$

$$\vec{r}(t) = \int_0^t \left( \frac{\vec{F}}{m} t + \vec{U}(0) \right) dt + \vec{r}(0)$$

$$\vec{r}(t) = \frac{1}{2} \frac{\vec{F}}{m} t^2 + \frac{\vec{F}}{m} \vec{U}(0) t + \vec{r}(0)$$

above is equation of motion at constant acceleration

$$\vec{a} = \frac{\vec{F}}{m}$$

with initial conditions $\vec{U}(0), \vec{r}(0)$

For gravity $\vec{F} = -mg \hat{e}_z$

$$\vec{r}(t) = -\frac{1}{2} mg t^2 \hat{e}_z + \vec{U}(0) t + \vec{r}(0)$$
perp: \[ N - F_g \cos \theta = 0 \quad \text{no acceleration in } \perp \]
\[ N = F_g \cos \theta = mg \cos \theta \quad \text{direction} \]

parallel: \[ F_g \sin \theta - F_f = ma \quad \text{friction} \]
\[ mg \sin \theta - \mu N = ma \quad F_f = \mu N \]
\[ mg \sin \theta - \mu mg \cos \theta = ma \]
\[ mg (\sin \theta - \mu \cos \theta) = ma \]
\[ a = g (\sin \theta - \mu \cos \theta) \]

For static case where \( a = 0 \), \( \mu = \mu_s \) coefficient of static friction and \( F_f^{\text{max}} = \mu_s N \), then
\[ \sin \theta_{\text{max}} - \mu_s \cos \theta_{\text{max}} = 0 \]
Determines max angle of incline \( \theta_{\text{max}} \) before block slip
\[ \tan \theta_{\text{max}} = \mu_s \]

For case where \( a > 0 \), \( \mu = \mu_k \) coefficient of kinetic friction
\[ a = g (\sin \theta - \mu_k \cos \theta) \]
In general, \( \mu_k < \mu_s \)
Linear restoring force

\[ F(x) = -kx \quad \text{Hook's law: stretched spring} \]

\[ F = ma \quad \Rightarrow \quad m \frac{d^2x}{dt^2} = -kx \]

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} x \]

Define \[ \omega_0 = \frac{k}{m} \]

\[ \frac{d^2x}{dt^2} = -\omega_0^2 x \quad \text{Simple harmonic oscillator} \]

General solution has the form

\[ x(t) = A \sin(\omega_0 t - \delta) \]

\[ = A \cos \delta \sin(\omega_0 t) - A \sin \delta \cos(\omega_0 t) \]

oscillations with angular freq \[ \omega_0 = \sqrt{\frac{k}{m}} \]

Note \[ F(x) = -kx = -\frac{d}{dx}(\frac{1}{2}kx^2) \]

\[ \Rightarrow F(x) \ \text{is conservative with potential} \quad U(x) = \frac{1}{2}kx^2 \]

Total mechanical energy

\[ E = T + U = \frac{1}{2} mv^2 + \frac{1}{2} kx^2 \quad \text{is conserved} \]
General Conserved Forces

in one dimension

$$-\int F \cdot dr^2 = U_2 - U_1$$

$$E = T + U = \frac{1}{2} m v^2 + U(x)$$ is conserved

$$\Rightarrow v^2 = \frac{2}{m} [E - U(x)]$$

$$\Rightarrow v = \frac{dx}{dt} = \pm \sqrt{\frac{2}{m} [E - U(x)]}$$

$$\Rightarrow \int dt = \int \frac{\pm dx}{\sqrt{\frac{2}{m} [E - U(x)]}}$$

$$t - t_0 = \int_{x_0}^{x} \frac{\pm dx}{\sqrt{\frac{2}{m} [E - U(x)]}}$$ where $x(t_0) = x_0$

If we know $U(x)$, we can in principle do the integration and get $t - t_0 = \Phi(x)$

To function we get after doing integral

we can then solve for $x(t) = \Phi^{-1}(t - t_0)$
In general we can understand what to expect for such motion.

Points $x_0$ and $x_1$ are stable equilibrium.

At these points, $F = -\frac{dU}{dx} = 0$

$\Rightarrow$ no force on particle, so particle at rest.

Also, if perturb particle's position $x_0 \Rightarrow x_0 + \delta$,
the resulting force $F$ pushes particle back to $x_0$.

Point $x_4$ is unstable equilibrium.

Here $F = -\frac{dU}{dx} = 0$ also, so particle at rest. But if perturb $x_4 \Rightarrow x_4 + \delta$, the resulting force pushes the particle away from $x_4$.

If the particle has energy $E_1$, it may either be at the stable points $x_1$ or $x_3$ or it may oscillate between points $x_2$ and $x_3$.

For such oscillation, $E = U$ and hence $T = 0$ and so $V = 0$, when $x = x_2$ or $x_3$. These are the "turning points" where velocity vanishes and particles motion reverses.

At $x = x_0$, $U$ is minimum and so $T \neq 0$ hence $V$.
If particle has energy $E_2$, it will come in from left, slow down at step when reaches $x_0$, then reverse directions and travel back to left.

Near a minimum of $U(x)$, say at $x_0$, one can always write

Taylor expansion: $U(x) \approx U(x_0) + \frac{1}{2} U''(x_0) (x-x_0)^2$

$\Rightarrow F = -\frac{du}{dx} = -U''(x_0) (x-x_0)$

$U'' = \frac{d^2U}{dx^2}$

Neumann's 2nd Law: Then

$m \frac{d^2x}{dt^2} = -U''(x_0) (x-x_0)$

Let $x' = x-x_0$, then $dx' = dx$

$m \frac{d^2x'}{dt^2} = -U''(x_0) x'$

Since $x_0$ is a minimum, $U''(x_0) > 0$

$\Rightarrow \frac{d^2x'}{dt^2} = -\frac{U''(x_0)}{m} x'$

$x'$ undergoes simple harmonic motion at angular frequency $\omega_0 = \sqrt{\frac{U''(x_0)}{m}}$

Curvature of $U(x)$ at minimum determines $\omega_0$.
velocity dependent force

A particle moving through a fluid (such as air) experiences a drag force that can often be approximated as

\[ F_d(v) = -k v^n \hat{v} \]

where:

- \( k \) is positive
- \( \hat{v} \) is in direction of \( v \)

Experimentally, for small objects moving at relatively low velocities (~ 2 m/s = 80 ft/s) in air, \( n = 1 \). For higher velocities (but lower than the speed of sound \( \approx 330 \text{ m/s} \approx 1100 \text{ ft/s} \) ), \( n = 2 \).

In the latter regime, for air we have

\[ F_d(v) = -\frac{1}{2} c_D \rho A v^2 \hat{v} \]

where:
- \( c_D \) is a dimensionless drag coefficient
- \( \rho \) is the density of air
- \( A \) is the cross section of object 1 to direction of motion

More generally, for air, a good approximate is

\[ F_d(v) \approx -(c_1 v + c_2 v^2) \hat{v} \]

where for spherical objects of diameter \( D \):
- \( c_1 = 1.55 \times 10^{-6} D \) and \( c_2 = 0.22 D^2 \) in MKS units
Example

0) vertical fall, low velocities $F_d \sim \nu \quad \rho g \gg c_2 \nu$

$$\frac{md\nu}{dt} = F = -mg - c_1 \nu$$

Separate variables

$$\int \frac{md\nu}{-mg - c_1 \nu} = \int dt$$

$$\nu_0 = \nu(t_0)$$

$$t = -\frac{m}{c_1} \left[ \ln \left( \frac{\nu + \frac{mg}{c_1}}{\nu_0 + \frac{mg}{c_1}} \right) \right] = -\frac{m}{c_1} \ln \left( \frac{\nu + \frac{mg}{c_1}}{\nu_0 + \frac{mg}{c_1}} \right)$$

Define $T = \frac{m}{c_1}$, solve for $\nu$ in terms of $t$

$$e^{-T\nu} = \frac{\nu + \frac{mg}{c_1}}{\nu_0 + \frac{mg}{c_1}}$$

$$\nu = \left( \frac{\nu_0 + \frac{mg}{c_1}}{e^{-T\nu}} \right) - \frac{mg}{c_1}$$

$$\nu = \left( \nu_0 + \frac{mg}{c_1} \right)e^{-T\nu} - \frac{mg}{c_1}$$

As $t \to \infty$, velocity approaches the value

$$\nu_\infty = -\frac{mg}{c_1} \quad \text{known as the terminal velocity}$$

For an object falling from rest, $\nu_0 = 0$

$$\nu = \nu_0 \left( 1 - e^{-t/T} \right)$$
For a small raindrop with \( D = 0.5 \text{ mm} \)

we get \( C_1 = (1.55 \times 10^{-4}) (0.5 \times 10^{-3}) = 0.775 \times 10^{-7} \)

\[
\begin{align*}
m &= \frac{4}{3} \pi \left( \frac{D}{2} \right)^3 \left( \frac{k \rho \alpha}{10^6} \right) = 6.54 \times 10^{-8} \text{ kg} \\
\frac{l}{g} &= 19/\text{cm}^3
\end{align*}
\]

\[
\begin{align*}
c &= \frac{m}{C_1} = \frac{6.54 \times 10^{-8}}{0.775 \times 10^{-4}} = 0.84 \text{ sec}
\end{align*}
\]

\[
\begin{align*}
v_0 &= \frac{mg}{C_1} = (0.84 \text{ sec}) (9.8 \text{ m/s}^2) = 8.3 \text{ m/s}
\end{align*}
\]

So raindrops reach terminal velocity of 8.3 m/s in less than one second!

Compare this to the speed of a raindrop in free fall.

\[
\begin{align*}
\text{Without air resistance, the speed } v(t) = -gt + v(0) \\
x(t) &= -\frac{1}{2} gt^2 + v(0)t + x(0)
\end{align*}
\]

If dropped from \( x(0) = h \) at \( v(0) = 0 \), then after \( t \) sec, it goes a distance \( d \) given by

\[
\begin{align*}
t &= \sqrt{\frac{2h}{g}} \\
\text{speed at the time is } |v(t)| = gt = \sqrt{2gh}
\end{align*}
\]

(can have gotten above by energy conservation)

\[
\begin{align*}
\frac{1}{2}mv^2 &= mg \cdot h \\
\implies v &= \sqrt{2gh}
\end{align*}
\]
For a rain drop falling from 2000 m, the terminal speed is

\[ \nu = \sqrt{2(9.8)(2000)} \text{ m/s} = 198 \text{ m/s} \]

\[ = 198 \left( \frac{\text{m}}{\text{s}} \right) \left( \frac{100 \text{ cm}}{\text{m}} \right) \left( \frac{\text{in}}{2.54 \text{ cm}} \right) \left( \frac{\text{ft}}{12 \text{ in}} \right) \left( \frac{\text{mi}}{5280 \text{ ft}} \right) \left( \frac{\text{hr}}{3600 \text{ s}} \right) \]

\[ = 450 \text{ mph} \]

\[ \nu_0 = \frac{mg}{C_1} \text{ terminal speed in m increases for larger raindrops or hailstones} \]

\( \text{v} \) terminal fall, higher velocities \( F_d \propto v^2 \) \( C_1 \ll C_2 \)

in general: \( \frac{mdv}{dt} = -mg + C_2 v^2 \)

where \( + \) is for falling \( - \) is for rising

since \( F_d \) is opposite
to \( v \)

\[ m \int \frac{dv}{v^2 - \frac{mg}{C_2}} = \int dt = t \]

\[ \frac{m}{C_2} \int \frac{dv}{v^2 - \frac{mg}{C_2}} = \int dx = x \]

look up integral in handbook

\[ t = \frac{v}{v_0} \left( \tanh \left( \frac{v_0}{v_0} \right) - \tanh \left( \frac{\nu}{v_0} \right) \right) \]

where \( \nu = \sqrt{\frac{mg}{C_2}} \) and \( \frac{v}{v_0} = \frac{v}{\nu} \)
for an object dropped from rest, \( v_0 = 0 \)

\[ t = -t \tan^{-1}(\frac{v}{v_0}) \]

\[ v(t) = v_\infty \tan(\frac{t}{\tau}) \]

\[ \tau = \sqrt{\frac{m}{C_2 g}} = \sqrt{\frac{0.6}{(0.22)(9.8)}} = 0.4677 \text{ s} \]

\[ v_\infty = \tau g = (0.67)(9.8) \text{ m/s} = 65 \text{ m/s} \]

(3) Projectile motion in air

For English, assume \( F_d = -k \cdot v \)
(although it is not really true)

\[ \text{horizontal:} \quad m \frac{d^2 x}{dt^2} = -k m \frac{dx}{dt} \]

\[ \text{vertical:} \quad m \frac{d^2 y}{dt^2} = -k m \frac{dy}{dt} - mg \]

Solve in horizontal direction

\[ \frac{d^x}{dt} = -k \cdot v_x \Rightarrow v_x(t) = v_{x0} e^{-kt} \]
\[ y = \frac{v_y}{g} x \]

Where \( v_{0x} = v_0 \cos \theta \)

Horizontal distance traveled from \( x(0) = 0 \)

\[
x(t) = \int_0^t \frac{dx}{v_x(t)} = \int_0^t \frac{v_{0x}}{v_x(t)} e^{-\frac{gt}{v_{0x}}} dt
\]

\[
= \frac{v_{0x}}{k} \left( 1 - e^{-\frac{kt}{v_{0x}}} \right)
\]

The range of the projectile is defined by solving for \( y(t) = 0 \) (can't do analytically - need perturbation or numerical solution - see text)

Find time \( t_f \) where \( y = 0 \) - this is time projectile hits from range, \( x(0) \) then \( x(t_f) \).

But note, although we can't easily do above calculation, we can say that the range can never be greater than \( \frac{v_{0x}}{k} \).

\[ \frac{\varepsilon}{k} \]

Compare this to case where there is no drag force.

Then \( v_x(t) = v_{0x} \) is constant

See Fig 2.8 of text for graph of more complete computation of range.