**Integer Quantum Hall Effect (IQHE)**

1985 Nobel prize to Klaus von Klitzing for exact discovery
(1998 Nobel prize to Laughlin, Störmer and Tsuei for fractional quantum Hall effect)

Refs: Kittel 8th ed pg 499
K. Huang - Statistical Physics, 2nd ed., pg 261
K. von Klitzing, Rev. Mod. Phys. 58, 519 (1986)
R.B. Laughlin, PRB 23, 5632 (1981)

Consider the geometry of the Hall effect

![Diagram showing the Hall effect](image)

Current confined to flow along \( z \) but results in component \( \vec{J} \) along \( \vec{E} \) along \( \vec{z} \)

In general, the current \( \vec{J} \) is not parallel to the total electric field \( \vec{E} \). We can formulate this in terms of a conductivity tensor \( \sigma \)

\[
\vec{J} = \sigma \cdot \vec{E}
\]

Considering just the \( xy \) plane:

\[
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix} =
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
\]
For a system with rotational invariance, \( \hat{\mathbf{E}} \) must satisfy some constraints on its components.

For \( \hat{\mathbf{E}}_1 = \begin{pmatrix} E \\ 0 \end{pmatrix} \) we have \( \hat{j}_1 = \hat{\mathbf{E}}_1 = \begin{pmatrix} \sigma_{xx} E \\ \sigma_{xy} E \end{pmatrix} \)

For \( \hat{\mathbf{E}}_2 = \begin{pmatrix} 0 \\ E \end{pmatrix} \) we have \( \hat{j}_2 = \hat{\mathbf{E}}_2 = \begin{pmatrix} \sigma_{xy} E \\ \sigma_{yy} E \end{pmatrix} \)

But \( \hat{\mathbf{E}}_2 \) is just a 90° rotation of \( \hat{\mathbf{E}}_1 \):

So we expect that \( \hat{j}_2 \) must be a 90° rotation of \( \hat{j}_1 \):

\[
\begin{pmatrix} j_{2x} \\ j_{2y} \end{pmatrix} = \begin{pmatrix} j_{1y} \\ -j_{1x} \end{pmatrix}
\]

\( \Rightarrow \sigma_{xy} = -\sigma_{yx} \) and \( \sigma_{yy} = \sigma_{xx} \)

So conductivity tensor must be of the form

\[
\hat{\mathbf{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{yx} & \sigma_{xx} \end{pmatrix}
\]

The resistivity tensor is the inverse of the conductivity tensor

\[
\hat{\mathbf{\rho}} = \hat{\mathbf{\sigma}}^{-1}
\]

so \( \hat{\mathbf{E}} = \hat{\mathbf{\rho}} \cdot \hat{\mathbf{j}} \)

\[
\hat{\mathbf{\rho}} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{pmatrix}
\]

with \( \rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \)

\( \rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} \)

check that this \( \rho_{xx} \) and \( \rho_{xy} \) give \( \hat{\mathbf{\rho}} \cdot \hat{\mathbf{\sigma}} = \hat{\mathbf{I}} \) identity tensor
From the above, we can see that some of our intuition, derived from the \( H=0 \) case, does not hold when \( H>0 \).

When \( H>0 \), \( \sigma \) is diagonal, \( \sigma_{xy}=0 \)

\[ \Rightarrow \sigma \text{ is diagonal and } f_{xx} = \frac{1}{\sigma_{xx}} \]

resistivity is inverse of conductivity.

If we have a system where \( \sigma_{xx}=0 \), then
\( f_{xx} \to \infty \). System is insulating with infinite resistivity, \( \sigma_{xx}=0 \Rightarrow \) system is dissipative (Joule heating, \( I^2R \) is huge).

When \( H>0 \) \( \sigma_{xy} \neq 0 \) \( f_{xx} \neq \frac{1}{\sigma_{xx}} \)

If we have a system with \( \sigma_{xx}=0 \) then
\( f_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} = 0 \) \( \text{ (not infinite as when } H=0 \text{)} \)

\( \vec{F} = \begin{pmatrix} 0 & f_{xy} \\ -f_{xy} & 0 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} f_{xy} f_y \\ -f_{xy} f_x \end{pmatrix} \)

and we have \( \vec{f} \cdot \vec{E} = 0 \)! Current is \( I \) electric field.

Consider an electron carrying this current.

The force on the electron is \( \vec{F} \times \vec{E} \), the velocity of the electron is \( \vec{v} \times \vec{F} \). So \( \vec{f} \cdot \vec{E} = 0 \Rightarrow \vec{F} \cdot \vec{v} = 0 \) no work done on electron. Current flows without dissipation.
So when $H = 0$, then $\sigma_{xx} = 0$ is a dissipative state, but when $H > 0$, then $\sigma_{xx} = 0$ is a dissipationless state.

In the quantum Hall effect, we will find that at certain discrete values of $H$, we have

\[
\begin{align*}
\sigma_{xx} &= j_{xx} = 0 \\
\sigma_{xy} &= -\frac{1}{B} j_{xy} = \frac{\nu e^2}{h}
\end{align*}
\]

where $\nu$ in the integer quantum Hall effect is an integer, while in the fractional quantum Hall effect, $\nu$ is a simple rational fraction.

The quantum Hall state thus describes a dissipationless current flow in which the Hall conductance $\sigma_{xy}$ is quantized in units of $e^2/h$.

The fractional quantum Hall effect depends on understanding the effects of the interactions between electrons—we will not discuss it.

The integer quantum Hall effect can be understood in terms of non-interacting electrons. This is the case we will consider.

Quantization of $\sigma_{xy}$ happens independent of details of the system—it happens for real, dirty, experimental systems! So it reflects a deep fact about quantum mechanics of the system.
Conductivity and resistivity tensors within Drude model

We can now reexamine our discussion of the Hall effect, as computed within the Drude model, in terms of the conductivity and resistivity tensors.

Drude's eqn of motion for momentum \( \vec{p} = m \vec{v} \)

\[
\frac{dp}{dt} = \vec{F} - \frac{e}{c} \vec{v} \times \vec{H} \quad \text{in steady state} \quad \frac{dp}{dt} = 0
\]

\[
\Rightarrow \vec{p} = \frac{e}{c} \vec{H} \quad \Rightarrow \vec{v} = \frac{m}{e} \vec{F}
\]

For the Hall geometry \( \vec{F} = -e \vec{E} - e \frac{\vec{v}}{c} \times \vec{H} \) so

\[
\vec{v} = -\frac{e}{m} \vec{E} - \frac{e}{mc} \vec{v} \times \vec{H}
\]

\[
\begin{pmatrix}
\frac{v_x}{v_y}
\end{pmatrix} = -\frac{e}{m} \begin{pmatrix}
\frac{E_x}{E_y}
\end{pmatrix} - \frac{eH}{mc} \begin{pmatrix}
\frac{v_y}{-v_x}
\end{pmatrix}
\]

\[
\omega_c = \frac{eH}{mc}
\]

\[
\begin{pmatrix}
\frac{v_x + \omega_c v_y}{v_y - \omega_c v_x}
\end{pmatrix} = -\frac{e}{m} \begin{pmatrix}
\frac{E_x}{E_y}
\end{pmatrix}
\]

\[
-\frac{m}{et} \begin{pmatrix}
1 & \omega_c \\
-w_c & 1
\end{pmatrix} \begin{pmatrix}
\frac{v_x}{v_y}
\end{pmatrix} = \begin{pmatrix}
\frac{E_x}{E_y}
\end{pmatrix}
\]

Current \( \vec{j} = -em \vec{v} \) so \( \vec{v} = -\frac{\vec{F}}{em} \)
\[
\frac{m}{ne^2c} \begin{pmatrix}
1 & \omega c^2 \\
\omega c^2 & 1
\end{pmatrix}
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
\mathbf{E}_x \\
\mathbf{E}_y
\end{pmatrix}
\]

\[
\Rightarrow \quad \mathbf{F} = \frac{1}{\sigma_0} \begin{pmatrix}
1 & \omega c^2 \\
\omega c^2 & 1
\end{pmatrix}
\quad \text{with} \quad \sigma_0 = \frac{ne^2c}{m}
\]

\[
\Rightarrow \quad \sigma = \sigma_0 \frac{1}{\mathbf{1} + (\omega c^2)^2} \begin{pmatrix}
1 & -\omega c^2 \\
\omega c^2 & 1
\end{pmatrix}
\]

This is same result as found in HW 1 for problem #1 if only look at one type of charge carrier, or problem #2 if set \(\omega \to 0\).

We can relate the Hall coefficient \(R_H\) to the resistivity tensor as follows:

For a geometry in which the current flows along \(x\), then \(j_y = 0\) and we define

\[
R_H = \frac{E_y}{j_x H}
\]

\[
\Rightarrow \quad \begin{pmatrix}
E_x \\
E_y
\end{pmatrix} = \begin{pmatrix}
\mathbf{f}_{xx} & \mathbf{f}_{xy} \\
-\mathbf{f}_{xy} & \mathbf{f}_{xx}
\end{pmatrix} \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\mathbf{f}_{xx} \dot{x} \\
-\mathbf{f}_{xy} \dot{x}
\end{pmatrix}
\]

\[
\Rightarrow \quad R_H = \frac{E_y}{j_x H} = \frac{-\mathbf{f}_{xy}}{j_x H} = \frac{-\mathbf{f}_{xy}}{H} = R_H
\]

\(R_H\) is just the off-diagonal element of \(\mathbf{F}\) divided by \(H\).
From above:

\[ f_{xx} = \frac{1}{\sigma_0} = \frac{m}{m e^2 c} \]
\[ f_{xy} = \frac{\omega_{c} \tau}{\sigma_0} = \frac{e H \tau}{m c} \frac{m}{m e^2 c} = \frac{H}{m e c} \]

(compare with \( R^+ = -\frac{f_{xy}}{H} = \frac{1}{m e c} \) we get back our earlier result for \( R^+ \))

So classically we expect

\[ f_{xy}(H) = \frac{H}{m e c} \text{ linear in } H \]
\[ f_{xx}(H) = \frac{m}{m e^2 c} \text{ indep of } H \]

Note that in the limit of no electron scattering, i.e. \( \tau \to \infty \),
then \( f_{xx} = 5_{xx} = 0 \), \( f_{xy} = \frac{H}{m e c} \), we have a

designationless state with \( \vec{f} \perp \vec{E} \).
Experimental results showing both the integer and the fractional quantum Hall effect.

Figure 1.2: Integer and fractional quantum Hall transport data showing the plateau regions in the Hall resistance $R_H$ and associated dips in the dissipative resistance $R$. The numbers indicate the Landau level filling factors at which various features occur. After ref. [15].
Now compare to what one sees experimentally for a two dimensional electron gas. An idealized sketch showing only the integer quantum Hall effect looks as follows:

1. $S_{xy}(H)$ consists of a series of plateaus at values $S_{xy} = h/e^2$ where $s$ is an integer. The width of the plateaus increase as $T$ decreases.

2. The plateaus are centered at the values of $H$ which would give $S_{xy} = h/e^2$ in the classical Drude model, i.e.,

$$S_{xy} = \frac{h}{e^2} = \frac{h}{me^2} \Rightarrow H = \frac{m \hbar c}{e} = \frac{m \Phi_0}{e} \text{ with } \Phi_0 = \frac{h}{e}$$

Recall our earlier discussion of Landau levels where we defined the flux quantum $\Phi_0 = \frac{h}{2e}$. This means therefore has dimension of magnetic flux, and $\Phi_0 = 2\Phi_0$. 


\[ \frac{H}{\Phi_0} \] was the degeneracy of a Landau level if we counted electrons with both spin up and spin down and ignored the coupling of electron spin to \( H \).

\[ \frac{H}{\Phi_0} \] is then the degeneracy of a Landau level if we consider only electrons of one fixed spin orientation (we all up or all down).

3. In the center of the plateaus, for some interval of \( H \) about \( H = m\Phi_0 / L \), we have

\[ S_{xx} = 0 \]

The IQHE states at \( H = m\Phi_0 / L \) are therefore dissipationsless current carrying states with quantized off diagonal conductivity,

\[ -\sigma_{xy} = \frac{1}{\rho_{xy}} = \alpha e^2 / h. \]

\[ \frac{\alpha}{4} \] has the units of conductance and is called the \"quantum of conductance.\"

Because \( \sigma_{xy} \) is quantized in integer units of \( h/e^2 \), the quantum Hall effect is now used as a standard for measuring conductance.
Why two dimensions is important

In a theoretical model, it is resistivity $\sigma x$ ad $\sigma y$ that one computes. These are the quantities that are intrinsic to the material and the physical situation. The total resistance $R_{xx}$, $R_{xy}$ are related to $\sigma x$ ad $\sigma y$ by sample geometry. However, while theory calculates $\sigma$, it is $R$ that is directly measured in experiment. So to convert from measured $R$ to prediction for $\sigma$, it would seem that one needs to know very precisely the correct geometrical factors for this conversion. Not true in two dimensions. This is important because the theory predicts that $\sigma y$ is quantized in units of $\mathbf{h}/\mathbf{e}c^2$, a integer, whereas it is $R_{xy}$ that is the directly measured quantity. We want to be able to test the theory with out worrying about the geometry! To illustrate this, consider a simple rectangular geometry

\[ \text{cross sectional area } A = Wd \]

For geometry in which current flows along $x$ and so $\frac{\partial I}{\partial y} = 0$, we have
\[
R_{xy} = \frac{V_y}{I_x} = \frac{\Delta E_y}{\Delta I_x} = \frac{E_y}{I_x} = \rho_{xy}
\]

So \[ R_{xy} = \rho_{xy} \] in 2D

\( R_y \) and \( \rho \) have the same units in 2D and there is no geometrical factor relating \( R_{xy} \) to \( \rho_{xy} \). Direct measurement of \( R_{xy} \) gives the quantified \( \rho_{xy} \).

Note the units of \( \rho_{xy} = \frac{k}{\text{sec}^2} \) in 2D have dimensions of
\[
\frac{k}{\text{sec}^2} = \frac{\text{energy} \cdot \text{sec}}{\text{energy} \cdot \text{length} \cdot \text{length}}
\]

while the units of \( R \) in 2D have dimensions of
\[
R = \frac{V}{I} = \frac{\text{energy/charge}}{\text{charge/sec}} = \frac{\text{energy} \cdot \text{sec}}{\text{energy} \cdot \text{sec} \cdot \text{sec}} = \frac{\text{sec}}{\text{energy} \cdot \text{length} \cdot \text{length}}
\]

so the units of total resistance = units of resistivity only in two dimensions.

The conductance tensor \( G = R^{-1} \) inverse of resistance tensor. In quantum Hall state with \( j_{xx} = \sigma_{xx} = 0 \),

\[
- \frac{1}{G_{xy}} = \frac{1}{R_{xy}} = \frac{1}{\rho_{xy}} = -\sigma_{xy}
\]

\[ G_{xy} = \sigma_{xy} \] Conductance and conductivity have the same units, and transverse conductance = transverse conductivity.
Aside: How does one make a two dimensional electron gas experimentally?

Semiconductor inversion layer

Simple picture

![Diagram of metal and semiconductor with voltage drop and electron potential energy](image)

Adjust $V_0$ so that the width of the potential well $\approx$ de Broglie wavelength of the electrons in the semiconductor $\Rightarrow$ motion in $\hat{z}$ direction is quantized into discrete levels. For proper choice of $V_0$, and low $T$, one can ensure that all electrons in the ground state lie in the lowest $\hat{z}$ level. Electrons remain free in $\hat{x}$ and $\hat{y}$ directions parallel to the interface.

When one applies $\hat{H} = \hat{H}_x + \hat{H}_y$, the energy levels are given by the Landau levels $\varepsilon_n = \hbar \omega_c (n + 1/2) + \text{const}$

[lower eigenvalue for motion along $\hat{z}$]
Questions

1. When $H = \frac{m \phi_0}{a}$, why is $\phi_{xy} = \frac{\hbar}{2 \varepsilon e^2}$ the classical Drude value, and why is $\phi_{xx} = 0$?

2. Why is $\phi_{xy} = \frac{\hbar}{2 \varepsilon e^2}$ constant for a range of $H$ about $H = \frac{m \phi_0}{a}$, i.e., why are there flat plateaus in $\phi_{xy}$?

Partial answer to 1:

Consider the Landau level structure of the eigenstates of electron motion in the $xy$ plane. We need now to also consider the interaction of intrinsic electron spin with the magnetic field.

For an electron in Landau level $n$ with spin $S = \pm 1$, the energy is:

$$E(n, S) = \hbar \omega_c (n + \frac{1}{2}) \pm \mu_0 H$$

with $\mu_0 = \frac{e \hbar}{2me}$ the Bohr magneton.

Now note that $\mu_0 H = \frac{e \hbar}{mc} \frac{\hbar}{2} = \frac{1}{2} \hbar \omega_c$.
so \[ E(n,s) = \begin{cases} \frac{\hbar \omega_e}{2} (n+1) & \text{for } s = +1 \\ \frac{\hbar \omega_e}{2} n & \text{for } s = -1 \end{cases} \]

The up electron energy of Landau level \( n \) coincides with the down electron energy of Landau level \( n+1 \). The degeneracy of each Landau level remains \( \frac{1}{2} \) with
\[ \Omega_0 = \frac{\hbar c}{2e} \]

In practice, for electrons in a real metal or semiconductor, the "mass" that enters their equation of motion is not the free electron mass \( m \), but rather some effective mass \( m^* \) that includes effects of the periodic conic potential in which the electron moves. It is this \( m^* \) that enters the \( \hbar \omega \) of the energy of the Landau levels, \( \hbar \omega = \frac{e^2}{m^*c} \). However, for the interaction of the intrinsic spin with \( H \), the Bohr magneton involves the free electron mass \( m \). Hence there is no overlap of \( n^* \) with \( (n+1)^* \) energy levels as there would be for a true free electron gas, i.e. now

\[ E(n,s) = \frac{\hbar e H}{m^* c} \left( n + \frac{1}{2} \right) \pm \frac{e H}{2mc} \]
In general $m^* \neq m$ so the Landau level for $S = \pm 1$ is split off from the Landau level for $S = -1$, with no overlap between them. The density of states then looks like

\[ g(E) \]

\[ \begin{array}{ccc}
    n=0 & n=1 & n=2 \\
    \downarrow & \downarrow & \downarrow \\
    \uparrow & \uparrow & \uparrow \\
    \end{array} \]

Now, since the spin states of each Landau level are split off from each other, the degeneracy of each $S$-function in the above $g(E)$ is half what it was when we ignore coupling of spin to $\mathbf{H}$.

\[ \rightarrow \text{degeneracy of each Landau level is now} \]

\[ \frac{\mathbf{H}}{2\Phi_0} = \frac{\mathbf{H}}{\Phi_0} \quad \text{with} \quad \Phi_0 = \frac{\hbar e}{\epsilon} \]

Return to partial answer to question (c)

When $\mathbf{H} = \frac{m\Phi_0}{\epsilon}$ we have $n = \frac{\mathbf{H}}{\Phi_0}$

\[ \rightarrow \text{we have exactly $S$ lowest Landau levels completely filled, and all higher Landau levels completely empty ($a$ is integer)} \]
There is this finite energy gap to the lowest unoccupied single particle energy level.

$\varepsilon_{F}$

When $\mu W, t Wc \gg k_B T$

and $\mu W, t Wc \gg$ other energy scales such as $k_B T$, where $W$ is the max phonon freq, or $\varepsilon_{F}/e$, the max energy absorbed by electron from $E$ field between collisions.

Then electrons in filled Landau levels cannot scatter into an unoccupied state in an empty Landau level. There are no empty states close enough in energy.

When electrons cannot scatter, $T \to \infty$, we have $\sigma_{xx} = g_{xx} = 0$ dissipationless current flow.

This explains why $g_{xx} = 0$ when $H = n \hbar \omega_0$ for integer $n$.!