Why two dimensions is important

In a theoretical model, it is resistivity $\rho_x$ and $\rho_y$ that are computed. These are the quantities that are intrinsic to the material and the physical situation. The total resistance $R_{xx}$, $R_{xy}$ are related to $\rho_x$ and $\rho_y$ by simple geometry. However, while theory calculates $\rho$, it is $R$ that is directly measured in experiment. So to convert from measured $R$ to prediction for $\rho$, it would seem that one needs to know very precisely the correct geometric factors for this conversion.

Not true in two dimensions. This is important because the theory predicts that $\rho_y$ is quantized in units of $\hbar/e^2$, a integer, whereas it is $R_y$ that is the directly measured quantity. We want to be able to trust the theory with out worrying about the geometry! To illustrate this, consider a simple rectangular geometry.

\[ \text{in 3D} \]

\[ \text{cross sectional area } A = Wd \]

For geometry in which current flows along $x$ and so $\rho_y = 0$, we have
\[
\begin{pmatrix}
    E_x \\
    E_y
\end{pmatrix} = 
\begin{pmatrix}
    f_{xx} & f_{xy} \\
    -f_{xy} & f_{xx}
\end{pmatrix} \begin{pmatrix}
    \dot{i_x} \\
    0
\end{pmatrix} = 
\begin{pmatrix}
    f_{xx} \dot{i_x} \\
    -f_{xy} \dot{i_x}
\end{pmatrix}
\]

\[
f_{xx} = \frac{E_x}{\dot{i_x}} \quad f_{xy} = -\frac{E_y}{\dot{i_x}}
\]

So then \[
R_{xx} = \frac{V_x}{I_x} = \frac{E_x L}{\dot{i_x} A} = f_{xx} \left( \frac{L}{A} \right)
\]

\[
R_{xy} = -\frac{V_y}{I_x} = -\frac{E_y W}{\dot{i_x} A} = f_{xy} \left( \frac{W}{A} \right)
\]

We see that to convert the measured resistance \( R_{xx}, R_{xy} \) to resistivities \( f_{xx}, f_{xy} \) we need the geometrical factors \( (L/A) \) and \( (W/A) \).

\[
\wedge 2D
\]

\[
\text{Now } \dot{i_x} \text{ is a "sheet current"} \quad \text{i.e. current per unit width}
\]

\[
\text{total current } I_x = \dot{i_x} \text{ W}
\]

So we have \[
R_{xx} = \frac{V_x}{I_x} = \frac{E_x L}{\dot{i_x} W} = f_{xx} \left( \frac{L}{W} \right)
\]
but
\[ R_{xy} = \frac{-V_y}{I_x} = \frac{-V_y}{\frac{W E_y}{f_x}} = \frac{-E_y}{f_x} = f_{xy} \]

So \[ R_{xy} = f_{xy} \] in 2D

\[ R_{xy} \] and \[ f_{xy} \] have the same units in 2D and there is no geometrical factor relating \[ R_{xy} \] to \[ f_{xy} \]. Direct measurement of \[ R_{xy} \] gives the quantified \[ f_{xy} \].

Note the units of \[ f_{xy} = \frac{1}{\text{sec}^2} \] in 2D have dimensions of
\[ \frac{1}{\text{sec}^2} = \frac{\text{energy} \cdot \text{sec}}{\text{energy} \cdot \text{length} \cdot \text{length}} \]

while the units of \[ R \] in 2D have dimensions of
\[ R = \frac{V}{I} = \frac{\text{energy} / \text{charge}}{\text{charge} / \text{sec}} = \frac{\text{energy} \cdot \text{sec}}{(\text{charge})^2} \]
\[ = \frac{\text{energy} \cdot \text{sec}}{\text{energy} \cdot \text{length} \cdot \text{length}} \]

so 
\[ \frac{1}{R^2} \] has units of total resistance = units of resistivity only in two dimensions.

The conductance tensor \[ G = R^{-1} \] inverse of resistance tensor. In quantum Hall state with \[ f_{xx} = G_{xx} = 0, \]

\[ -G_{xy} = \frac{1}{R_{xy}} = \frac{1}{f_{xy}} = -f_{xy} \]

\[ G_{xy} = f_{xy} \] Conductance and conductivity have the same units, and transverse conductance = transverse conductivity.
But why in this case should $f_{xy}$ have its classical Drude value

$$\frac{h}{m_c} = \frac{m \phi_0}{e m_c} = \frac{k}{2e^2}?$$

Why can't quantum effects lead to something different for $f_{xy}$?

For the case of a completely filled Landau level, one can directly compute quantum mechanically the current $j_x$ and $j_y$. For $\mathbf{H} = H_0 \hat{\mathbf{z}}$ and total $\mathbf{E} = E \hat{\mathbf{y}}$

$$j_y = -\frac{e}{m} \sum_{n, s, k_x} \sum_{k_x} \langle \psi_{nk_x} | \frac{\hbar}{2} \frac{\partial}{\partial y} | \psi_{nk_x} \rangle$$

$$j_x = -\frac{e}{m} \sum_{n, s, k_x} \sum_{k_x} \langle \psi_{nk_x} | \frac{\hbar}{2} \frac{\partial}{\partial x} - \frac{e}{c} H_y | \psi_{nk_x} \rangle$$

where $\psi_{nk_x} = e^{i k_x x} \Phi_n (y-y_0)$ with

$$y_0 = \frac{1}{\omega_c} \int \frac{dx}{m} \left[ \frac{\hbar k_x}{m} - \frac{eE}{H} \right]$$

is the eigenfunction for the eigenstate in Landau level $n$, with eigenvalue $k_x$ (look back at Landau level notes).

The velocity operator is $\frac{\hbar}{i} \nabla - \frac{e}{c} A$

$$m \mathbf{V_x}^{op} = \frac{\hbar}{2} \frac{\partial}{\partial x} - \frac{e}{c} H_y \quad \text{using } A = x H_y$$

$$m \mathbf{V_y}^{op} = \frac{\hbar}{2} \frac{\partial}{\partial y}$$
The sum on $k_x$ is over all allowed values of $k_x$ from 0 to $k_{x\text{max}} = \frac{E}{\hbar v}$, and the sum on $n,s$ is over all filled Landau levels ($n$ integer, $s = \pm 1$ for spin up and down).

$\phi_n$ is the $n$th harmonic oscillator eigenstate.

You will do this for homework!

You will find $j_y = 0$ and $j_x = -\frac{mc e E}{\hbar}$.

Since $E = E_g$, this gives

\[ j = 8 \cdot \frac{e}{v} \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} \begin{pmatrix} 0 \\ E \end{pmatrix} = \begin{pmatrix} \sigma_{xy} E \\ \sigma_{xx} E \end{pmatrix} \]

\[ j_y = 0 \Rightarrow \sigma_{xx} = \boxed{f_{xx} = 0} \]

\[ j_x = -\frac{mc e E}{\hbar} \Rightarrow \sigma_{xy} = -\frac{1}{j_x} = -\frac{mc e}{\hbar} \]

\[ \Rightarrow f_{xy} = \frac{\hbar}{mc e} = \frac{m \phi_0}{\Delta} = \frac{\hbar}{2e^2} \]

So this calculation confirms the observed values of $f_{xx}$ and $f_{xy}$ at $H = \frac{m \phi_0}{\Delta}$. 
Note: We can only do the above calculation of \( f_x \) and \( f_y \) for the special case of completely filled Landau levels or no partially filled Landau levels.

In this case, since there is no electron scattering, the occupation function is just its equilibrium value, which at low \( k_B T \ll \hbar \omega \) is just 1 for a state in the filled Landau level, and 0 for a state in an empty Landau level.

If we had a partially filled Landau level, the applied fields would cause electrons to scatter (scattering now is allowed in a partially filled Landau level since there are degenerate unoccupied states) and this would create some non-equilibrium steady state probability distribution for describing the probability that a given state in the Landau level is occupied or not. We would then need some theory of how to compute this non-equilibrium distribution in order to compute the average \( f_x \) and \( f_y \).

But we still have not answered question (2). Why are there plateaus in \( f_x \) and \( f_y \)?
Why are there these plateaus?

We might expect that for $H = \hbar \phi_0 / \pi$ there is a partially filled Landau level. There are empty states right at the Fermi energy. \( \Rightarrow \) Scattering can take place \( \Rightarrow g_{xx} \neq 0 \) and no reason for \( g_{xy} \) to remain fixed at \( \hbar / e^2 \).

It turns out that to explain the plateaus we need to assume that the electron gas sees a random potential due to static impurities in the material. It is perhaps paradoxical that to observe best the quantization of conductance \( e^2 / h \) we need a dirty system and not a pure one, but this turns out to be true.

To explain the effect we need to discuss the phenomenon of localization of electron wavefunction in the presence of a random potential. Perturbation theory would suggest that when a random potential is added to the system, the degenerate Landau level states will split into an energy band of finite width. (In general, perturbations that obey no particular symmetry will split degenerate energy levels.) But in this case the effect of the random potential is even more dramatic. Not only
does it split the degenerate states of a given Landau level into a band of finite width in energy, but the states in the low and high energy tails of this band become localized; states in the middle of the band remain extended.

Our old eigenstates of the pure problem (no random potential) were

$$\Psi_{ik} \sim e^{ikx} \phi_n(y - y_0(x))$$

Such states are extended because there is a finite probability $|\Psi|^2$ to find the electron at any position $x$ in the system. The $e^{ikx}$ part travels through the entire system.

Impurities, however, create local potential wells that can trap electrons in bound states

$$\Psi(x) \sim e^{-|x - x_0|/\ell}$$
in which the electron is localized to position $x_0$ with a localization length scale $\ell$. 
In the presence of a random potential, the density of states of the 2D electron gas in a uniform magnetic field $\mathbf{B}$ will look like:

$$q(E)$$

- Extended states in the middle of band
- Localized states in tails of band

Now, as $H$ increases or decreases from $m\phi_0/$s,
the Fermi energy $E_F$ (which lies in the gap between the Landau level bands at $H = m\phi_0/$s) first enters the region of localized states. Localized states cannot carry any current because the electron is bound to a particular impurity site and not free to travel throughout the sample (at least this is true at low temperature). $\Rightarrow$ $j_x$ and $j_y$ do not change from the values they had at $H = m\phi_0/$s.

As $E_F$ is fixed experimentally at $j_x$ does not change.

- handwritten note: $\mathcal{F}$
If $E$ is held fixed experimentally, and $\tilde{\rho}$ does not change as it is varied, then by

$$\tilde{f} = \tilde{\rho} \cdot \tilde{E} \Rightarrow \tilde{\rho} \text{ does not change}$$

$$\Rightarrow \text{ plateaus in } f_{xx} \text{ and } f_{xy}$$

Problem: Above argument shows why there are plateaus in $f_{xx}$ and $f_{xy}$. But now that we have added impurities and the degenerate Landau level states have changed into a band of finite energy width with both extended and localized states, how do we know that we still have $f_{xx} = 0$ and $f_{xy} = \frac{\hbar^2}{2e^2}$? That calculation was done for the pure system!

Answer: Gauge invariance argument due to Laughlin shows that $f_{xx} = 0$ and $f_{xy} = \frac{\hbar}{2e^2}$ even in the random case.