If an electron could travel a distance in \( k \)-space larger than \( \Delta k \) size in between collisions, then a DC \( \overline{E} \) field would produce oscillating current! However, collisions will spoil this effect, and an electron in general will have only small changes in \( \overline{k} \) before it gets scattered and its \( \overline{E} \) randomized.

However, the fact that \( \Delta k \ll \overline{k} \) near band max produces the phenomenon of holes - metal can behave as if it had positive carriers.

Consider 1-d example. Near band minima at \( k_0 \), we can expand \( \varepsilon(k) \approx \varepsilon(k_0) + \varepsilon''(k_0) (k-k_0)^2 \)

where \( \varepsilon''(k_0) \equiv \frac{\hbar^2}{m^*} > 0 \)

we call \( m^* \) the effective mass of electrons at band minimum.

Then semiclassical equations are:

\[
\frac{\dot{\overline{k}}}{\hbar} = \frac{\delta \varepsilon(k)}{\delta \overline{\mathbf{k}}} = \frac{1}{\hbar} \frac{\partial \varepsilon(k)}{\partial k} = \frac{\hbar^2}{m^*} \left( \varepsilon''(k_0) \right) \overline{k} = \frac{\hbar^2}{m^*} \overline{k} \quad \text{(just like classical particle of mass } m^* \text{, charge } e) \]

\[
\frac{\dot{\overline{p}}}{\hbar} = -e [\overline{E} + \left( \frac{\hbar}{m^*} \right) \overline{k} \times \overline{H}] \quad \text{momentum} \]

\[
\Rightarrow m^* \dot{\overline{p}} = -e [\overline{E} + \left( \frac{\hbar}{m^*} \right) \overline{k} \times \overline{H}] \quad \text{momentum} \]

So electrons near band minima behave like classical electron of charge \(-e\) and mass \( m^* = \frac{\hbar^2}{\frac{\partial^2 \varepsilon}{\partial k^2}} \).
But for an electron near top of band, we expand

\[ E(k) = E(k_1) + \frac{d^2E}{dk^2} (k_1) \left( \frac{k - k_1}{2} \right)^2 \]

where we define \[ \frac{d^2E}{dk^2} = -\frac{\hbar^2}{m^*_h} \quad \text{where } m^*_h > 0 \]

Now \[ V(k) = \frac{-\hbar^2}{m^*} \frac{k - k_1}{m^*} \Rightarrow m^*_h \dot{v} = -\hbar k \]

So \[ \hbar k = -e \left[ E + \frac{\mathbf{v} \times \mathbf{H}}{c} \right] \Rightarrow m^*_h \dot{v} = +e \left[ E + \frac{\mathbf{v} \times \mathbf{H}}{c} \right] \]

Electron near top of band behaves like classical particle of mass \[ m^*_h = -\frac{\hbar^2}{2d^2E/dk^2} \] and charge \[ +e \] like a positive charge.

This is referred to as a hole.

In three dimensions, if max and min of band occur at point of cubic symmetry, we still can expand

\[ E(k) = E(k_0) + \frac{\hbar^2}{2m^*_h} \left( k - k_0 \right)^2 \]

to define effective mass. However if no symmetry, then we need to define effective mass tensor

\[ M_{ij} \]

\[ M_{ij}^{-1} \]

\[ M_{ij}^{-1} k_j = 0 \]

\[ M_{ij}^{-1} = \pm \frac{1}{\hbar^2} \frac{\partial^2 E}{\partial k_i \partial k_j} \]

Equation of motion will be

\[ M \cdot \dot{\mathbf{v}} = -e \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{H}}{c} \right) \]
in most generality, away from max or min, can write

\[ \frac{\partial \text{ } E}{\partial t} = \frac{1}{\hbar} \left( \frac{1}{2} \frac{\partial E}{\partial \mathbf{r}} \right) = \frac{1}{\hbar} \frac{\partial E}{\partial \mathbf{k}} \frac{\partial \mathbf{k}}{\partial t} \]

define

\[ M^\dagger(k) = \frac{1}{\hbar^2} \frac{\partial E}{\partial \mathbf{k}} \]

\[ M^\star(k) \mathbf{r} = e \left[ \frac{\mathbf{\dot{r}}}{E} + \mathbf{v}(\mathbf{r}) \times \mathbf{H} \right] \]

\[ \text{ if } \frac{\partial E}{\partial \mathbf{k}} \neq 0 \]

So states near top of band

behave like (+) particles of mass \( m^* \)

To compute current in a partially full band, note

\[ \frac{\mathbf{j}}{e} = -e \int \frac{d^3 \mathbf{k}}{4\pi^3} \mathbf{\dot{v}}_n(\mathbf{k}) = -e \left[ -\int \frac{d^3 \mathbf{k}}{4\pi^3} \mathbf{\dot{v}}_n(\mathbf{k}) \right] \]

occupied states

\[ = e \int \frac{d^3 \mathbf{k}}{4\pi^3} \mathbf{\dot{v}}_n(\mathbf{k}) \]

unoccupied states

since \( \int \mathbf{\dot{r}} + \int \mathbf{\dot{u}} = 0 \)

So we can regard electric current as due to either the

occupied electric states (with

So current due to electrons in occupied states is the same

as current that would be if there levels were

empty and the unoccupied states were filled with particles of charge +e.
Note: Unoccupied states evolve under some equations of motion as occupied states — as if they were filled with electrons of charge $-e$.

But unoccupied states generally lie near top of band, so they evolve in time like classical particles of charge $+e$!

For a given band, we can choose to describe it in either the electron or hole picture, but not both.

For a band mostly empty convenient to describe as classical electrons of charge $-e$ and mass $m^* = \frac{\hbar^2}{(\frac{d^2E}{d^2k})_{min}}$

For a band mostly full convenient to describe as classical particles (holes) of charge $+e$ and mass $m^* = -\frac{\hbar^2}{(\frac{d^2E}{d^2k})_{max}}$

Very useful for describing semiconductors.
Motion in Uniform Magnetic Field

\[ \frac{\partial \mathbf{k}}{\partial t} = -e \frac{\mathbf{v} \times \mathbf{B}}{\hbar} \]

For motion in uniform field, \( \mathbf{E}(\mathbf{r}(t)) = \frac{d\mathbf{E}}{dt} = \nabla \mathbf{V} \cdot \mathbf{k} = 0 \)

since \( \nabla \mathbf{V} = \mathbf{v} \times \mathbf{H} \):

so electron moves on surface of constant energy,

also \( \frac{d}{dt} (\mathbf{k} \cdot \mathbf{H}) = \mathbf{k} \cdot \mathbf{H} = 0 \) as \( \mathbf{H} \cdot (\mathbf{v} \times \mathbf{H}) = 0 \)

\( \Rightarrow \) electrons move on curves formed by intersection of plane of constant \( \mathbf{H} \) (take it in \( \mathbf{k} || \mathbf{H} \)) with surfaces of constant energy.

For spherical energy surface

Sence of orbit: since \( \mathbf{v} = \frac{1}{m} \frac{d\mathbf{r}}{dt} \) points from lower \( \mathbf{B} \) to higher \( \mathbf{B} \)

If \( \mathbf{H} \) is up, one walks \( \mathbf{k} \) orbit so that higher energy states are on right as \( \mathbf{k} \times \mathbf{H} \to \mathbf{H} \times \mathbf{v} \)

(hole orbit).

In closed orbits, If surface encloses region of higher energy, direction is opposite

\( \Rightarrow \) if surface encloses lower energy (electron orbit)

\( \Rightarrow \) 3-d curve, \( \mathbf{H} \parallel \mathbf{k} \) so in nearly free electron approx.

\( \mathbf{k} \) holes in 1st band

 electons in 2nd band
The real space orbits \( \mathbf{r}(t) \) can be found:

\[
\mathbf{r}_\perp(t) = \mathbf{r} - \hat{\mathbf{H}}(\mathbf{r} \cdot \hat{\mathbf{H}})
\]

Position in plane \( \perp \) to \( \hat{\mathbf{H}} \)

\[
\hat{\mathbf{H}} \times \frac{\mathbf{v} \times \hat{\mathbf{H}}}{c} = -\frac{e}{c} \mathbf{H} x (\mathbf{v} \times \hat{\mathbf{H}}) = -\frac{e}{c} \mathbf{H} (\mathbf{v} - \hat{\mathbf{H}} \cdot \mathbf{v})
\]

using \( \mathbf{v} = \frac{\mathbf{v}}{c} \)

and vector identity

\[
= -\frac{e}{c} \mathbf{r}_\perp
\]

so \( \mathbf{r}_\perp(t) - \mathbf{r}_\perp(0) = -\frac{tc}{eH} \mathbf{H} \times (\mathbf{r}(t) - \mathbf{r}(0)) \)

So \( \mathbf{r}_\perp \) orbit is just \( \mathbf{r} \) orbit rotated by \( 90^\circ \) about \( \hat{\mathbf{H}} \),

and scaled by \( \frac{tc}{eH} \)

In \( \parallel \) direction

\[
\mathbf{r}_\parallel(t) = \mathbf{r}_\parallel(0) + \int_0^t \mathbf{v}_\parallel(t) \, dt = \mathbf{r}_\parallel(0) + \int_0^t \frac{1}{k_\parallel} \frac{\partial E}{\partial k_\parallel} \, dt
\]

Need not be uniform in \( t \) as \( \frac{\partial E}{\partial k_\parallel} \) can vary as \( k_\parallel \) varies.

For spherical energy surface, we get classical result:

Electron moves in circular orbit \( \perp \) to \( \mathbf{H} \).

However, energy surfaces need not be spherical

- (when they get too near zone boundaries) - need
  not be closed curves! See figure 12.8 in text

\[
\text{When orbits are open, applying \( \mathbf{H} \) can lead to}
\]