Motion in uniform $\vec{E}$ and $\vec{H}$ fields

Hall effect and magnetoresistance

\[ \vec{t} \dot{\vec{r}} = -e \left[ \vec{E} + \frac{\vec{E} \cdot \vec{E}}{c^2} \times \vec{H} \right] \]

\[ \Rightarrow \vec{H} \times \vec{t} \dot{\vec{r}} = -e \vec{H} \times \vec{E} - \frac{eH}{c} \vec{E} \]

\[ \vec{E} \perp = -\frac{te}{eH} \vec{H} \times \vec{t} \dot{r} + \vec{w} \]

\[ \vec{w} = \frac{cE}{H} (\vec{E} \times \vec{H}) \]

Motion is as before, but with drift velocity $\vec{w}$ added.

To determine orbits in $k$ space mode:

\[ \vec{t} \dot{\vec{k}} = -e \vec{E} - \frac{e}{c} \frac{1}{k} \frac{\partial E}{\partial k} \times \vec{H} \]

Write $\vec{E} = - (\vec{E} \times \vec{H}) \times \vec{H}$

True when $\vec{E} \perp \vec{H}$

\[ \Rightarrow \vec{E} = - \frac{e}{c} \frac{\partial E}{\partial k} \times \vec{H} \]

Same as if $\vec{E}$ was absent and band structure replaced by

\[ E(k) = E(\vec{k}) - \frac{k\vec{k} \cdot \vec{w}}{c} \]

Orbits are intersections of surfaces of constant $E$ with planes $\perp \vec{H}$

We will assume that $-\frac{k\vec{k} \cdot \vec{w}}{c}$ small enough so that $E$ the constant $E(k)$ surface is closed (open) so is the constant $\vec{E}(k)$ surface.  Good approx in most cases - see text for estimate of numbers.
In nearly free electron model

\[ \varepsilon(k^2) = \frac{\hbar^2 k^2}{2m} \]

Surface of constant energy \( \varepsilon \) is sphere of radius

\[ \sqrt{\frac{2m \varepsilon}{\hbar^2}} = k \] in \( k \)-space

\[ \varepsilon(k^2) = \frac{\hbar^2 k^2}{2m} - \hbar \bar{w} \cdot \vec{k} \]

Surface of constant \( \varepsilon \) is given by

\[ \frac{\hbar^2}{2m} (k - \frac{\bar{w}}{\hbar})^2 = \varepsilon + \frac{1}{2} m \bar{w}^2 \]

Sphere in \( k \)-space of radius

\[ k = \sqrt{\frac{2m}{\hbar^2} (\varepsilon + \frac{1}{2} m \bar{w}^2)} \]

centered about \( \bar{k}_0 = \frac{\bar{w}}{\hbar} \)

Surface of constant \( \varepsilon \) is shifted by \( \frac{\bar{w}}{\hbar} \) term in direction \( \bar{w} \)
Hall effect: \[
\mathbf{j} = -\frac{e}{c} \mathbf{v} \times \mathbf{B} + \mathbf{w}, \quad \mathbf{w} = \frac{eE}{H} (\mathbf{E} \times \mathbf{B})
\]

Current in plane \( \perp \) to \( \mathbf{H} \) is

\[
\mathbf{j} = n_e e \mathbf{c} \mathbf{v} - n_e \mathbf{c} \langle \mathbf{v}_\perp \rangle\]

where \( \langle \mathbf{v}_\perp \rangle \) is steady state average over all occupied electron orbits and over collisions.

\[
\mathbf{j} = -ne \mathbf{c} \mathbf{v} + ne \mathbf{c} \mathbf{E} \times \langle \mathbf{c} \rangle 
\]

Case (1) All occupied (or unoccupied) orbits are closed.

Then for large enough \( H \) so that \( \omega_c \tau \gg 1 \)
(where \( \tau \) is collision time, and \( \omega_c = \frac{eH}{m^*c} \)), electron makes many periods of its closed orbits between successive collisions.

We can estimate \( \langle \mathbf{c} \rangle \) in this large \( H \) case as follows:

Averaging over electron motion between two successive collisions at \( t=0 \) and \( t=t_0 \) we get

\[
\langle \mathbf{c} \rangle = \frac{1}{t_0} \int_0^{t_0} \mathbf{c}(t) \, dt = \frac{\mathbf{c}(t_0) - \mathbf{c}(0)}{t_0}
\]

where \( \mathbf{c}(0) \) is wave vector of electron as it emerges from the first collision at \( t=0 \), and \( \mathbf{c}(t_0) \) is wave vector of electron just before second collision at \( t=t_0 \).

As in the Drude model, we may assume that electrons emerge from a collision with an equilibrium distribution determined by the local temperature and chemical potential. Since the Fermi distribution

\[
f(E(k)) = \frac{1}{\exp \left( \frac{E(k)}{k_T} \right) + 1}
\]

depends on \( k \) only via energy \( E(k) \), and \( E(k) = E(-k) \),
we have, after averaging over the electron energy, from the collision at \( t = 0 \), \( \langle \hat{\mathbf{r}}(0) \rangle = 0 \). So \( \langle \hat{\mathbf{r}} \rangle = \hat{\mathbf{r}}(t_0)/t_0 \).

We now average over the time until the second collision, \( \langle t_0 \rangle = T \) (the time is distributed randomly with average equal to \( T \)). Since \( \omega_c T \gg 1 \), the electron makes many orbits between collisions, \( \hat{\mathbf{r}}(t_0) \) when averaged over collision time \( t_0 \), is equally likely to lie anywhere along the closed orbit.

\[ \Rightarrow \langle \hat{\mathbf{r}}(t_0) \rangle = \text{ (average } \hat{\mathbf{r}} \text{ on orbit)} \].

If electric field \( \mathbf{E} \neq 0 \), then (average \( \hat{\mathbf{r}} \) on orbit) = 0 also. But when \( E \neq 0 \), (average \( \hat{\mathbf{r}} \) on orbit) \( \sim m^* \hat{\mathbf{w}} / \kappa \). To see this, use effective mass approximation,

\[ e^* (k) = \frac{k^2 \rho^2}{2m^*} \]

Then orbit lies on curve of constant

\[ E(k) = E(k) - \mathbf{k}, \hat{\mathbf{w}} \text{, which lies on sphere centered at } \hat{\mathbf{k}}_0 = \frac{m^* \hat{\mathbf{w}}}{\kappa} \]. So (average \( \hat{\mathbf{r}} \) on orbit) = \( \langle \hat{\mathbf{r}}(t_0) \rangle = \hat{\mathbf{k}}_0 \)

\[ \Rightarrow \langle \hat{\mathbf{r}} \rangle = \left( \frac{\langle \hat{\mathbf{r}}(t_0) \rangle}{t} \right) = \frac{\hat{\mathbf{k}}_0}{T} = \frac{m^* \hat{\mathbf{w}}}{\kappa T} \].

So contribution of \( \langle \hat{\mathbf{r}} \rangle \) term to current is

\[ \frac{n e^* c}{\hbar} \hat{\mathbf{x}} \frac{m^* \hat{\mathbf{w}}}{\kappa T} = \frac{n e}{\omega_c} \hat{\mathbf{x}} \hat{\mathbf{w}} \]

smaller than drift contribution to current

\[ \hat{\mathbf{j}} = -n \mathbf{e} \hat{\mathbf{v}} \text{ by a factor } \frac{1}{\omega_c} \ll 1 \]

So \( \hat{\mathbf{j}} = -n \mathbf{e} \hat{\mathbf{v}} \) given just by drift velocity \( \hat{\mathbf{w}} \) in high field limit.
In this case \( \vec{j} \) is \( \perp \) to \( \vec{H} \) \( \Rightarrow \) \( \vec{j} \) is \( \perp \) to \( \vec{E} \) and \( \vec{H} \)

\( \Rightarrow \) Lorentz force so strong that electrons move \( \perp \) to \( \vec{E} \) and do not acquire any energy from the \( \vec{E} \)-field.

The Hall coefficient in this limit is just \( \frac{E}{j \cdot H} \)

\[ R_{H \rightarrow \infty} = \frac{E}{j \cdot H} \]

but \( \vec{E} = \frac{cE}{H} (\vec{E} \times \vec{H}) \)

\( \Rightarrow \)

\[ R_{H \rightarrow \infty} = \frac{E}{-n_e c E H} = -\frac{1}{n_e c} \]

\( \text{(Drude value)} \)

The above was for closed occupied orbitals.

If we had closed unoccupied orbitals we would use the hole picture to get

\[ R_{H \rightarrow \infty} = \frac{1}{n_h \cdot c} > 0 \]

\( (n_h \) is density of holes, each hole has charge \( +e \) )

If there is more than one partially full band with only closed occupied or unoccupied orbitals then

\( \vec{j} = -n_{\text{eff}} \frac{e}{c} (\vec{E} \times \vec{H}) \) where \( n_{\text{eff}} = n - n_h \)

\[ R_{H \rightarrow \infty} = -\frac{1}{n_{\text{eff}} \cdot c} \]

The effects of holes explains why \( R_0 \) can have non-Drude values, and even be \( > 0 \).
See text for what happens when $\text{Meff} = 0$. This is the case for undoped semiconductor.

Another way to view things is to do it in terms of conductivity tensor. Keeping contribution to $\mathbf{j}$ from the $\mathbf{k}$-ten gives

$$\mathbf{j} = -ne\mathbf{v} + \frac{ne}{\text{Meff}} \mathbf{A} \times \mathbf{v}, \quad \mathbf{v} = \frac{\mathbf{E} \times \mathbf{A}}{H}$$

for $\mathbf{A} = \mathbf{z}$ direction we have

$$\mathbf{j} = \frac{neC}{H} (\mathbf{z} \times \mathbf{E} + \frac{1}{\text{Meff}} \mathbf{z} \cdot \mathbf{E}) = \mathbf{\sigma} \cdot \mathbf{E}$$

with $\mathbf{\sigma} = \frac{neC}{H} \begin{pmatrix} \frac{1}{\text{Meff}} & -1 \\ 1 & \frac{1}{\text{Meff}} \end{pmatrix}$

or writing $\frac{\sigma_0}{\text{Meff}} = \frac{ne^2C}{m^*eHc} \frac{m^*e}{eHc} = \frac{neC}{H}$ where

$\sigma_0$ Dunde conductivity $\Rightarrow \mathbf{\sigma} = \sigma_0 \begin{pmatrix} \frac{1}{(\text{Meff})^2} & -1 \\ 1 & \frac{1}{(\text{Meff})^2} \end{pmatrix}$

$\sigma_0$ (compare with prob #1 on HW #1!!)

$\Rightarrow$ resistivity tensor $\mathbf{\rho} = \mathbf{\sigma}^{-1} = \frac{1}{\sigma_0} \begin{pmatrix} \frac{1}{(\text{Meff})^2} & -1 \\ 1 & \frac{1}{(\text{Meff})^2} \end{pmatrix}$

$\Rightarrow$ bulk coefficient $\rho = \rho_{xx} = \frac{\rho_{yy}}{\rho_{zz}}$ as $(\text{Meff}) < 1$

$$\mathbf{\rho} = \frac{1/\sigma_0}{1 + \frac{1}{(\text{Meff})^2}} \begin{pmatrix} 1 & -\text{Meff} \text{Meff} \\ -\text{Meff} & 1 \end{pmatrix} \Rightarrow \frac{1}{\sigma_0} \begin{pmatrix} 1 & -\text{Meff} \\ -\text{Meff} & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}$$
\[ \mathbf{f} = \sigma \mathbf{E} \]

\[ \sigma = \frac{\sigma_0}{\omega_c T} \begin{pmatrix} \frac{1}{\omega_c T} & -1 \\ 1 & -\frac{1}{\omega_c T} \end{pmatrix} \]

\[ \sigma_0 = \frac{ne^2 c}{m^*} \]

\[ \omega_c T = \frac{eH}{mc} \gg 1 \]

Then \[ \mathbf{E} = \mathbf{p} \cdot \mathbf{f} \]

where \[ \mathbf{p} = \frac{1}{\sigma_0} \begin{pmatrix} -\omega_c T & 1 \\ -1 & \omega_c T \end{pmatrix} \begin{pmatrix} p_x x \\ p_y y \end{pmatrix} = \begin{pmatrix} p_x x + p_y y \\ p_x y - p_y x \end{pmatrix} \]

For \[ \mathbf{f} = \mathbf{j} x \] then \[ \mathbf{E}_y = p_x x \mathbf{j} = -p_y y \mathbf{j} \]

Hall coeff: \[ R = \frac{E_y}{j H} = \frac{-p_y y}{\sigma_0 H} = -\frac{\omega_c T}{m^* c} \frac{m^*}{ne^2 c} \frac{1}{H} \]

\[ = -\frac{1}{m^* c} \text{ (true value)} \]

For holes, \[ \mathbf{j} = m^* \mathbf{w} \]

For electrons we used \[ \mathbf{j} = -me \mathbf{w} + me \mathbf{H} \times \mathbf{w} \]

For holes we use instead \[ \mathbf{j} = +m^* \mathbf{w} - m^* \mathbf{H} \times \mathbf{w} \]

Since charge carriers have charge \( e \).

All results carry through except take \( e \rightarrow -e \)

\[ \Rightarrow R = \frac{1}{m^* c} \]