From Coulomb to Maxwell

Electrodynamics is concerned with one particular attribute of matter—charge.

Experimentally, it was observed that certain bodies exert long-range forces on each other that are certainly not gravitational—they are not proportional to the mass and they can be repulsive as well as attractive. The source of this new force was defined to be the “charge” of the object.

Electrostatics

Coulomb’s Law: for charge \( q_1 \) at \( \vec{r}_1 \) and charge \( q_2 \) at \( \vec{r}_2 \), if separation \( |\vec{r}_2 - \vec{r}_1| \) is much greater than the size of either charge, then

\[
\vec{F}_{21} = \frac{k q_1 q_2}{r_{12}^2} \hat{r}_{12}
\]

force on 2 due to 1

central force—point from 1 to 2

inverse square law
$k_1$ is a universal constant of nature that determines the strength of the force when $g$ is expressed in terms of some arbitrary reference charge.

Since we only know about charge by measuring the Coulomb force, we are in principle free to choose $k_1$ to be anything we like - our choice then determines the units that charge is measured in.

In the MKS system of units (same as SI system) charge is measured in the historical unit, the "coulomb." Then $k_1$ has the value $k_1 = \frac{1}{4\pi\varepsilon_0} = 10^{-7} \text{ C}^2$, where $c$ is speed of light in a vacuum. The units of $k_1$ are $\text{N} \cdot \text{m}^2 / \text{coul}^2$.

In the CGS system of units (also called esu - electrostatic units) one fixes $k_1 = 1$ and charge is measured in "statcoulombs." $k_1$ is taken dimensionless, so statcoulomb = $(\text{N} \cdot \text{m}^2)^{1/2}$.

Another reasonable modern choice would be to measure charge in integer multiples of the electron charge. This would yield a different value for $k_1$.

In this class we will be using CGS units. But we keep $k_1$ general for now.
Superposition

For charges $q_i$ at positions $\vec{r}_i$, the force on charge $q_i$ at position $\vec{r}$ is

$$\vec{F} = k \frac{q_i}{c^2} \sum \frac{q_i}{|r_i - r|^3}$$

forces add linearly

Conservation of charge

charge is neither created nor destroyed

$$\frac{d}{dt} \sum q_i = 0$$

where sum is over all charges in system

Continuum charge density

for charges $q_i$ at positions $\vec{r}_i$, define,

$$\rho(\vec{r}) = \sum q_i \delta(\vec{r} - \vec{r}_i)$$

$\delta(r_i - r_j)$ is Dirac $\delta$-function with properties:

$$\int_V d^3r \delta(\vec{r} - \vec{r}_i) = \begin{cases} 1 & \text{if } \vec{r}_i \in V \\ 0 & \text{otherwise} \end{cases}$$

$$\int_V d^3r f(\vec{r}) \delta(\vec{r} - \vec{r}_i) = \begin{cases} f(\vec{r}_i) & \text{if } \vec{r}_i \in V \\ 0 & \text{otherwise} \end{cases}$$

for any scalar function $f(\vec{r})$
\[
\frac{1}{V} \int \delta(r) = \frac{1}{c} \delta \int d^3r \cdot \delta(r-r_c)
\]

\[
= \text{the total charge enclosed by volume } V
\]

\[\Rightarrow \delta \text{ has units of charge per volume}\]

\[\Rightarrow \delta(r) \text{ has units of } 1/\text{vol}\]

\[
\mathbf{E} = k_e \int d^3r' \cdot \delta(r') \cdot \frac{r-r'}{|r-r'|^3}
\]

We will often forget that \(\delta(r)\) is in principle made up of a distribution of point charges, and take it to be a smooth continuous function.

Charge conservation: \[
\frac{d}{dt} \int d^3r \cdot \delta(r) = 0
\]

assuming \(V\) is so big that it contains all the charge, and no charge flows through the surface of \(V\)
Electric Field

$E(\vec{r})$ is the force per unit charge that would be felt by an infinitesimal test charge $q_0$ at position $\vec{r}$.

$$E(\vec{r}) = \frac{1}{q_0} \vec{F} = k_1 \int \frac{\sigma(\vec{r'})}{|\vec{r} - \vec{r'}|^3} d^3r' \quad (\ast)$$

In principle, the above is solution to all electrostatic problems. In practice, we may not always know $\sigma(\vec{r})$, but may need to solve for it self consistently with $E$. It will help to have another formulation of the above in terms of differential equations. We get these by taking the divergence and curl of eq (\ast) above to get (see proof later).

$$\nabla \cdot \vec{E} = 4\pi k_1 \rho \quad (1) \quad \text{Gauss' Law}$$

$$\nabla \times \vec{E} = 0 \quad (2) \quad \text{true only in statics!}$$

The proof of the above will follow on next page.

We also can recast (1) and (2) in integral form as follows by Gauss' Theorem

$$\int_S d^3r \nabla \cdot \vec{E} = \oint_S d\vec{a} \cdot \vec{E} = 4\pi k_1 \int_V d^3\rho$$

Total charge enclosed in $V$

by Stokes

$$\int_S d\vec{a} \cdot \nabla \times \vec{E} = \oint_C d\vec{l} \cdot \vec{E} = 0$$
As above, \( \vec{n} \) is outward pointing normal to surface \( S \) \\
\( \vec{t} \) is differential tangent to curve \( C \) bounding surface \( S \)

**Proofs** that (1) and (2) follow from (x)

First note that

\[
\frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} = -\vec{n} \left( \frac{1}{|\vec{r}' - \vec{r}|} \right)
\]

To see this, let \( \vec{r} = \vec{r}_1 \), and do the calculation in spherical coords centered at \( \vec{r} = 0 \).

\[
\frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} = \frac{\vec{n}}{r^2}
\]

\[-\vec{n} \left( \frac{1}{|\vec{r}' - \vec{r}|} \right) = -\vec{n} \frac{1}{r} \]

Since \( \vec{r} = \vec{r}' - \vec{r} \), we have \( \vec{n} \) (which differentiates with respect to \( \vec{r} \))

is the same as \( \vec{n} \frac{1}{r} \) (which differentiates with respect to \( \vec{r} \)).

So \[-\vec{n} \left( \frac{1}{r} \right) = \vec{n} \frac{d}{dr} \left( \frac{1}{r} \right) \] using \( \vec{n} \) in spherical coords

\[
= -\vec{n} \frac{1}{r^2}
\]

So from Coulomb we have

\[
\vec{E}(\vec{r}) = k_1 \int \frac{\vec{d}^3 r'}{|r - r'|^3} \vec{r} - \vec{r}'
\]

\[
= -\vec{V} \left( k_1 \int \frac{\vec{d}^3 r'}{|r - r'|} \right) \hspace{1cm} \vec{E} \text{ is gradient of scalar function}
\]

\[
\Rightarrow \vec{\nabla} \times \vec{E} = 0 \hspace{1cm} \text{since the curl of a gradient always vanishes}
\]

\[
\vec{\nabla} \times \vec{\nabla} \phi = 0 \hspace{1cm} \text{for any scalar function} \phi
\]
\[ \nabla \cdot \vec{E} = -\nabla^2 \left( r_1 \int d^{3-1} \frac{\rho(r')}{|r-r'|} \right) \]

where \( \nabla^2 = \nabla \cdot \nabla \)

Consider

\[ \nabla^2 \left( \frac{1}{|r-r'|} \right) \]

as before, define \( \vec{r}' = \vec{r} - \vec{r}' \), so \( \vec{\nabla}' = \vec{\nabla} \), and go to spherical coords centered at \( \vec{r}' = 0 \).

\[ \nabla^2 \left( \frac{1}{|r-r'|} \right) = \nabla^2 \left( \frac{1}{r} \right) \]

use expression for \( \nabla^2 \)

in spherical coords

\[ \frac{d^2}{dr^2} r \left( \frac{1}{r} \right) \]

= \( 0 \) \ for \( r \neq 0 \)

\[ \text{Singular \ at \ } r = 0 \]

So

\[ \nabla^2 \left( \frac{1}{r} \right) \]

vanishes everywhere except at \( r = 0 \)

to see what happens at \( r = 0 \), consider integrating over a sphere \( \rho \) of radius \( R \) centered at the origin

\[ \int_{d^3r} \nabla^2 \left( \frac{1}{r} \right) = \int_{d^3r} \nabla \cdot \nabla \left( \frac{1}{r} \right) = \oint_{\text{surface } S} \mathbf{\hat{n}} \cdot \nabla \left( \frac{1}{r} \right) \]

using Gaussian Theorem

\[ \int_{d^3r} \nabla^2 \left( \frac{1}{r} \right) = \int_{d^3r} \frac{d^2}{dr^2} \left( \frac{1}{r} \right) = -\frac{1}{r^2} \]

is constant on surface \( S \) so

\[ \oint_{\text{surface } S} \mathbf{\hat{n}} \cdot \nabla \left( \frac{1}{r} \right) = 4\pi R^2 \left( -\frac{1}{r^2} \right) = -4\pi \]
Above was integrating over a sphere, but we would get same result if integrated over any volume containing \( \vec{r} = 0 \).

\[ S \] is sphere of radius \( R \)

\[ S' \] is any surface

Let \( V' \) be volume between \( S \) and \( S' \).

Then by Gauss theorem

\[
\int_{V'} d^3r \left( \nabla \cdot \frac{\vec{V}}{r} \right) = \oint_{S} \text{d}a \, \hat{n} \cdot \frac{\vec{V}}{r} - \oint_{S'} \text{d}a \, \hat{n} \cdot \frac{\vec{V}}{r}
\]

\[=0 \quad \text{since} \quad \nabla^2 \left( \frac{1}{r} \right) = 0 \quad \text{everywhere in} \quad V'
\]

\[
\Rightarrow \oint_{S} \text{d}a \, \hat{n} \cdot \frac{\vec{V}}{r} = \oint_{S'} \text{d}a \, \hat{n} \cdot \frac{\vec{V}}{r}
\]

\[
\Rightarrow \int_{V} d^3r \, \nabla^2 \left( \frac{1}{r} \right) = \int_{V} d^3r \, \nabla^2 \left( \frac{1}{r} \right)
\]

\[=\text{bounded by} \quad S' \quad \Rightarrow \quad \text{bounded by} \quad S
\]

So we conclude:

\[
\int_{V} d^3r \, \nabla^2 \left( \frac{1}{r} \right) = \begin{cases} -4\pi & \text{if} & \vec{r} = 0 \text{ in } V \\ 0 & \text{if} & \vec{r} = 0 \text{ not in } V \end{cases}
\]

\[
\Rightarrow \nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(\vec{r}) \quad \text{Dirac delta function}
\]

\[
\nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta \left( \vec{r} - \vec{r}' \right)
\]
So now
\[ \nabla \cdot \vec{E} = -k_1 \int d^3r' \ p(r') \ \nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \]

\[ -k_1 \int d^3r' \ p(r') \ (-4\pi) \delta(\vec{r} - \vec{r}') \]

\[ = 4\pi k_1 \rho(\vec{r}) \quad \text{by property of \( \delta \) function} \]

proof is done!

we have shown that
\[ \vec{E}(\vec{r}) = k_1 \int d^3r' \ p(\vec{r}') \ \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \Rightarrow \begin{cases} \nabla \cdot \vec{E} = 4\pi k_1 \rho \\ \nabla \times \vec{E} = 0 \end{cases} \]

is the reverse true? is it the formulation in terms of partial differential equations completely equivalent to Coulomb's law? yes! because of Helmholtz's Theorem.

Helmholtz Theorem of vector calculus — if one specifies the divergence and curl of a vector function, and boundary conditions (here \( E \to 0 \) as \( r \to \infty \) and one is away from all charges), then vector function is uniquely determined.