Maxwell's equations apply exactly to the true microscopic electric and magnetic fields that arise from all charges and currents.

\[ \nabla \cdot \mathbf{b} = 0, \quad \nabla \times \mathbf{e} + \frac{1}{c} \frac{\partial \mathbf{b}}{\partial t} = 0 \]

\[ \nabla \cdot \mathbf{e} = 4\pi \rho, \quad \nabla \times \mathbf{b} = \frac{4\pi}{\varepsilon} \mathbf{j}_0 + \frac{\partial \mathbf{e}}{\partial t} \]

where \( \mathbf{e} \) and \( \mathbf{b} \) are microscopic fields from total charge density \( \rho \) and current density \( \mathbf{j}_0 \).

However, in most problems involving macroscopic objects, if we took \( \rho \) and \( \mathbf{j}_0 \) to describe charge and current of each individual atom in a material, then they, and the resulting \( \mathbf{e} \) and \( \mathbf{b} \), would be enormously complicated functions varying rapidly over distances \( \sim 10^{-8} \) cm and times \( \sim 10^{-16} \) sec.

In classical E&M we are generally concerned with phenomena that vary extremely slowly compared to these length and time scales.
Rather than worry about the microscopic details of $f$ and $g$ as results $E$ and $\bar{E}$ we want to describe phenomena in terms of averaged quantities that are smoothly varying at the atomic scale. This results in what are known as the macroscopic Maxwell equations.

Dielectric Materials

A dielectric material is an insulator. Electrons are bound to the ionic cores of the atoms. When no electric field is present, the averaged $\bar{p}$ in the dielectric vanishes! One might therefore think that electrodynamics in a dielectric is just due to whatever "extra" or "free" charge is added to the dielectric. However, this is not true due to the phenomena of "polarization."

When $E = 0$:
- Electron cloud centered on ionic core
- Dipole moment vanishes

When $E > 0$:
- Electron cloud and ionic core displaced $\bar{d} \times \vec{E}$
- Atomic polarizability
- Has dipole moment $\bar{p} = \bar{q} \bar{d} \times \vec{E}$
Polarization density \( \vec{P}(\vec{r}) = \sum_i \vec{p}_i \delta(\vec{r} - \vec{r}_i) \)

dipole moment of atom \( i \) at position \( \vec{r}_i \)

Polarization density \( \vec{P} \) can give rise to regions of net charge - sometimes called "bound charge"

**Example**

\[ \vec{\mathcal{E}} = 0 \quad \text{uniform } \mathcal{E} \rightarrow \mathcal{E} > 0 \]

For a non-uniform \( \mathcal{E} \), atoms are more strongly polarized where \( \mathcal{E} \) is largest.

For uniform \( \vec{P} \), build up surface charge \( \sigma_b \)

For nonuniform \( \vec{P} \), also can build up vol charge density \( \rho_b \).
We now carry out the averaging explicitly to see how such polarization enters the macroscopic Maxwell equations (Jackson 6.6).

Define spatially averaged quantities by

$$\langle F(\vec{r},t) \rangle = \int d^3r' \, f(\vec{r}') \, F(\vec{r} - \vec{r}',t)$$

where $f(\vec{r})$ vanishes for $|\vec{r}|$ large on microscopic length scales, but short on macroscopic length scales. $f(\vec{r})$ normalized to unity: $\int d^3r \, f(\vec{r}) = 1$.

Other details of $f(\vec{r})$ are not too important, as long as $f(\vec{r})$ is a smooth function of $\vec{r}$.

\[ f \approx 1 \text{ for } r < R \]
\[ f \approx 0 \text{ for } r > 2R \]

where $R$ is length scale in between micro + macro.

\[ \frac{\partial}{\partial r_c} \langle F(\vec{r},t) \rangle = \int d^3r' \, f(\vec{r}') \, \frac{\partial F(r - r',t)}{\partial r_c} = \langle \frac{\partial F}{\partial r_c} \rangle \]

\[ \frac{\partial}{\partial t} \langle F(\vec{r},t) \rangle = \langle \frac{\partial F}{\partial t} \rangle \]

Define the macroscopic fields

\[ \overrightarrow{E}(\vec{r},t) = \langle \overrightarrow{e}(\vec{r},t) \rangle \]
\[ \overrightarrow{B}(\vec{r},t) = \langle \overrightarrow{b}(\vec{r},t) \rangle \]
\[ \text{Then} \quad \vec{\nabla} \cdot \vec{b} = 0 \quad \Rightarrow \quad <\vec{\nabla} \cdot \vec{b}> = 0 \]
\[ \Rightarrow \quad \vec{\nabla} \cdot <\vec{b}> = 0 \]
\[ \Rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0 \]
\[ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \Rightarrow \quad \vec{\nabla} \times <\vec{E}> + \frac{2}{\text{at}} <\vec{B}> = 0 \]
\[ \Rightarrow \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \]

Rearranging Maxwell's equations upon averaging, become:

\[ \vec{\nabla} \cdot \vec{E} = 4\pi <f_0> \]
\[ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} <\vec{f}_0> + \frac{2}{c} \vec{E} \]

Consider \(<f_0>\)

\[ f_0 = \sum_i \delta (\vec{r} - \vec{r}_i(t)) \quad \text{sum over all charges} \]

Consider dividing the charge into "free" charges and "bound" charges, where the latter are associated with the molecules that make up the dielectric.

\[ f_{\text{free}} = \sum_{i \text{ free}} \delta (\vec{r} - \vec{r}_i(t)) \quad \text{sum over only free charges} \]

\[ f_{\text{bound}} = \sum_n \rho_n (\vec{r}, t) \quad \text{charge distribution of molecule } n \]

\[ \rho_n = \sum_{i \text{ on } n} \delta (\vec{r} - \vec{r}_i(t)) \quad \text{sum over charges on molecule } n \]
\[
\langle \hat{f}_n(\vec{r},t) \rangle = \sum_{\text{cen}} g_i \int d^3r' \ f(\vec{r}') \ \rho_n(\vec{r} - \vec{r}',t) \\
= \sum_{\text{cen}} g_i \int d^3r' \ f(\vec{r}') \ \delta(\vec{r} - \vec{r}_i(t)) \\
= \sum_{\text{cen}} g_i \ f(\vec{r} - \vec{r}_i(t)) \\
\] 

Write \( \vec{r}_i(t) = \vec{r}_n(t) + \vec{r}_{ni}(t) \) 

The position of the center of mass of molecule \( n \) with respect to the center of mass of molecule \( n \) 

\[
\langle \hat{f}_n(\vec{r},t) \rangle = \sum_{\text{cen}} g_i \ f(\vec{r} - \vec{r}_n - \vec{r}_{ni}) \\
\] 

Since the \( \vec{r}_{ni} \) are all of atomic length scale, and \( \vec{r} \) a slowly varying on the length scale, we can expand 

\[
\langle \hat{f}_n(\vec{r},t) \rangle = \sum_{\text{cen}} g_i \left[ f(\vec{r} - \vec{r}_n) - \left( \nabla f(\vec{r} - \vec{r}_n) \right) \cdot \vec{r}_{ni} \\
+ \frac{1}{2} \sum_{\alpha \beta} \frac{\partial f(\vec{r} - \vec{r}_n)}{\partial r_\alpha} (\vec{r}_{ni})_\alpha (\vec{r}_{ni})_\beta + \cdots \right] \\
= f(\vec{r} - \vec{r}_n) \sum_{\text{cen}} g_i \\
- (\nabla f(\vec{r} - \vec{r}_n)) \cdot \sum_{\text{cen}} g_i \vec{r}_{ni} \\
+ \sum_{\alpha, \beta=1}^3 \left( \frac{1}{6} \frac{\partial f(\vec{r} - \vec{r}_n)}{\partial r_\alpha} \right) \sum_{\text{cen}} g_i (\vec{r}_{ni})_\alpha (\vec{r}_{ni})_\beta 
\]
Define \( q_n = \sum_{i \in n} q_i \) total charge molecule \( n \)

\[ \hat{p}_n = \sum_{i \in n} q_i \hat{r}_{ni} \] dipole moment about center of mass of molecule \( n \)

\[ \hat{Q}'_n = \sum_{i \in n} 3 q_i \hat{r}_{ni} \hat{r}_{ni} \] quadrupole moment about center of mass of molecule \( n \)

(Prime on \( \hat{Q}' \) since definition here is a little different from that of multipole exp)

\[ \langle \Phi_n (\vec{r}, t) \rangle = q_n f(\vec{r} - \vec{r}_n) - \hat{p}_n \cdot \nabla f(\vec{r} - \vec{r}_n) \]

\[ + \frac{1}{6} \sum_{\alpha \beta} \left( \hat{Q}'_n \right)_{\alpha \beta} \frac{2}{\partial x_\alpha \partial x_\beta} f(\vec{r} - \vec{r}_n) \]

Now use \( \langle \delta (\vec{r} - \vec{r}_n) \rangle = f(\vec{r} - \vec{r}_n) \) by definition of avragin

\[ \Rightarrow \langle \Phi_n (\vec{r}, t) \rangle = \langle q_n \delta (\vec{r} - \vec{r}_n) \rangle \]

\[ - \nabla \cdot \langle \hat{p}_n \delta (\vec{r} - \vec{r}_n) \rangle \]

\[ + \frac{1}{6} \sum_{\alpha \beta} \hat{Q}'_n \frac{2}{\partial x_\alpha \partial x_\beta} \langle \delta (\vec{r} - \vec{r}_n) \rangle \]

Now

\[ \langle \Phi_{\text{bound}} (\vec{r}, t) \rangle = \sum_n \langle \Phi_n (\vec{r}, t) \rangle \]

\[ = \langle \sum_n q_n \delta (\vec{r} - \vec{r}_n) \rangle - \nabla \cdot \langle \sum_n \hat{p}_n \delta (\vec{r} - \vec{r}_n) \rangle \]

\[ + \frac{1}{6} \sum_{\alpha \beta} \hat{Q}'_n \frac{2}{\partial x_\alpha \partial x_\beta} \langle \sum_n (\hat{Q}'_n)_{\alpha \beta} \delta (\vec{r} - \vec{r}_n) \rangle \]
Define \( \vec{P}(r, t) \equiv \left\langle \sum_n \vec{p}_n \delta(r - \vec{r}_n) \right\rangle \) average polarization density

\( \vec{Q}'(r, t) \equiv \frac{1}{6} \left\langle \sum_n \vec{Q}_n \delta(r - \vec{r}_n) \right\rangle \) average quadrupole density

\[
\langle \mathbf{P}_{\text{bound}} \rangle = \sum_n \left\langle q_n \delta(r - \vec{r}_n) \right\rangle - \vec{\nabla} \cdot \vec{P} + \sum_{\alpha \beta} \frac{e^2}{\omega_{\alpha \beta} \sigma_{\alpha \beta}} \vec{Q}_{\alpha \beta}
\]

Define the macroscopic charge density

\( \rho \equiv \left\langle \sum_{\text{free}} q \delta(r - \vec{r}_i) \right\rangle + \left\langle \sum_{\text{molec}} q_n \delta(r - \vec{r}_n) \right\rangle \)

Then

\[
\vec{\nabla} \cdot \vec{E} = \langle \rho_0 \rangle = 4\pi \left[ \rho - \vec{\nabla} \cdot \vec{P} + \sum_{\alpha \beta} \frac{e^2}{\omega_{\alpha \beta} \sigma_{\alpha \beta}} \vec{Q}_{\alpha \beta} \right]
\]

\[
\sum_{\alpha} \frac{2}{\sigma_{\alpha \alpha}} \left[ E_{\alpha} + 4\pi P_{\alpha} - \frac{1}{\sigma_{\alpha \alpha}} \sum_{\beta} \frac{e^2}{\sigma_{\beta \beta}} \vec{Q}_{\alpha \beta} \right] = 4\pi \rho
\]

Define electric displacement vector

\( \mathbf{D}_\alpha = E_{\alpha} + 4\pi P_{\alpha} - \sum_{\beta} \frac{2}{\sigma_{\beta \beta}} \vec{Q}_{\alpha \beta} \)

Then

\( \vec{\nabla} \cdot \mathbf{D} = 4\pi \rho \)

In most materials, the quadrupole and higher terms are negligible and we can take

\( \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \)
Why quadrupole + higher order terms can generally be ignored:

Let $a_0$ be the length scale that characterizes the size of a molecule in the dielectric.
Let $l$ be the typical spacing between molecules.
Let $L$ be the length scale of the spectral averaging function $f(r)$. $L \gg a_0$

Dipole moment $p = a_0$

$\Rightarrow$ polarization $P \sim \frac{a_0}{L}$

$\nabla \cdot \overrightarrow{P} \sim \frac{(a_0/L)^2}{L^3}$ since $P$ cannot vary on length scale short than the averaging length $L$

Quadrupole moment:

$q^2 \sim a_0^2$

Quadrupole density $Q \sim \frac{a_0^2}{L^3}$

$\frac{\partial^2 Q}{\partial r \partial r_p} \sim \frac{(a_0)^2}{L} \frac{1}{L^3}$

Each higher moment gives extra factor $a_0$.
Each higher derivative gives extra factor $\frac{1}{L}$.

So quadrupole is smaller than dipole term by factor $(\frac{a_0}{L})^2 \ll 1$. Higher terms smaller by additional factors of $(\frac{a_0}{L})$. 
For insulators we generally have $q_n = 0$.

In this case the macroscopic $q$ is just the free charge $q = \langle q_{\text{free}} \rangle$. And the bound charge is just $\langle q_{\text{bound}} \rangle = -\nabla \cdot \mathbf{P}$.

At a surface of a dielectric:

\[
\text{area} \rightarrow \begin{array}{c}
\pm \text{Gaussian pillbox } V \\
\text{width } l \rightarrow 0 \\
\vec{m} \text{ normal}
\end{array}
\]

\[
- \int_V d^3r \nabla \cdot \mathbf{P} = - \int_S d\mathbf{a} \hat{\mathbf{m}} \cdot \mathbf{P}
\]

Contrib from sides $\rightarrow 0$ as $l \rightarrow 0$.

Contrib from outside surface $= 0$ as $P = 0$ outside.

\[
= \hat{\mathbf{m}} \cdot \mathbf{P} d\mathbf{a}
\]

Only contrib is from inside surface.

\[
= \int_V d^3r q_{\text{bound}}
\]

($\vec{m}$ is outward normal).

As $l \rightarrow 0$, $\int_V d^3r q_{\text{bound}} \rightarrow \int_S d\mathbf{a} \sigma_{\text{bound}} = d\mathbf{a} \sigma_{\text{bound}}$ surface charge.

\[
\Rightarrow \sigma_{\text{bound}} = \hat{\mathbf{m}} \cdot \mathbf{P}
\]

at surface of dielectric.